Statistical behaviour of optical vortex fields

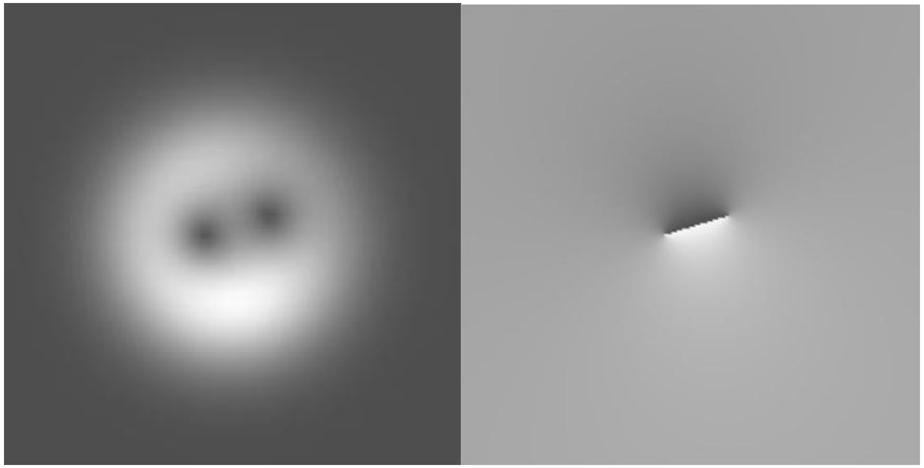
F. Stef Roux

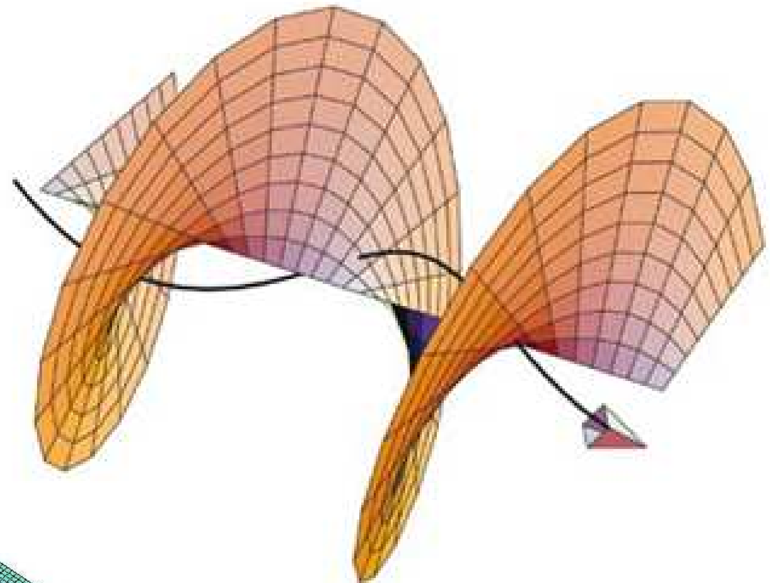
CSIR National Laser Centre, South Africa

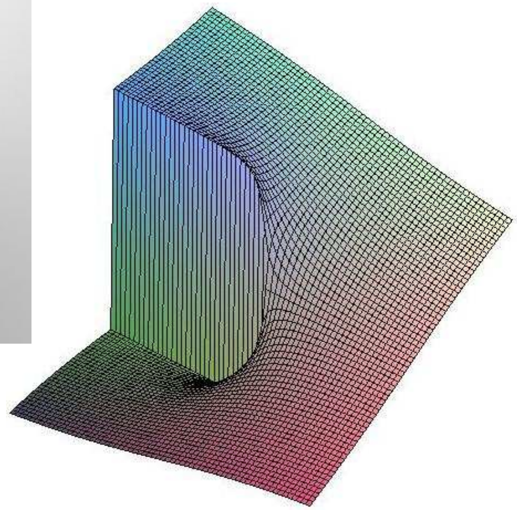
Colloquium presented at
School of Physics
National University of Ireland
Galway, Ireland
21 September 2009

our future through science

What are optical vortices?



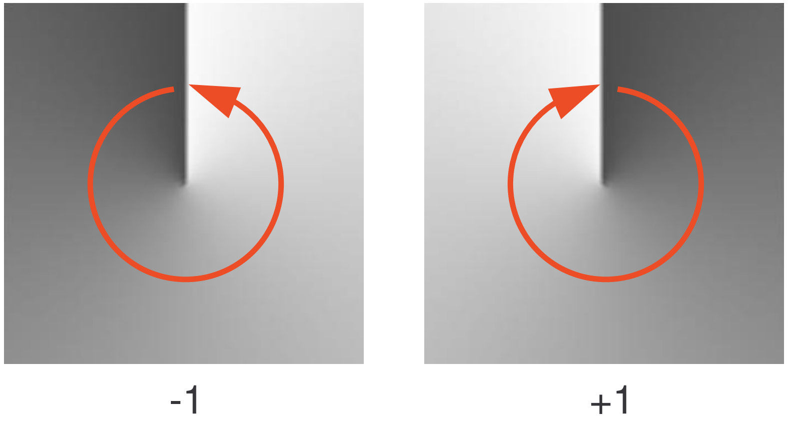


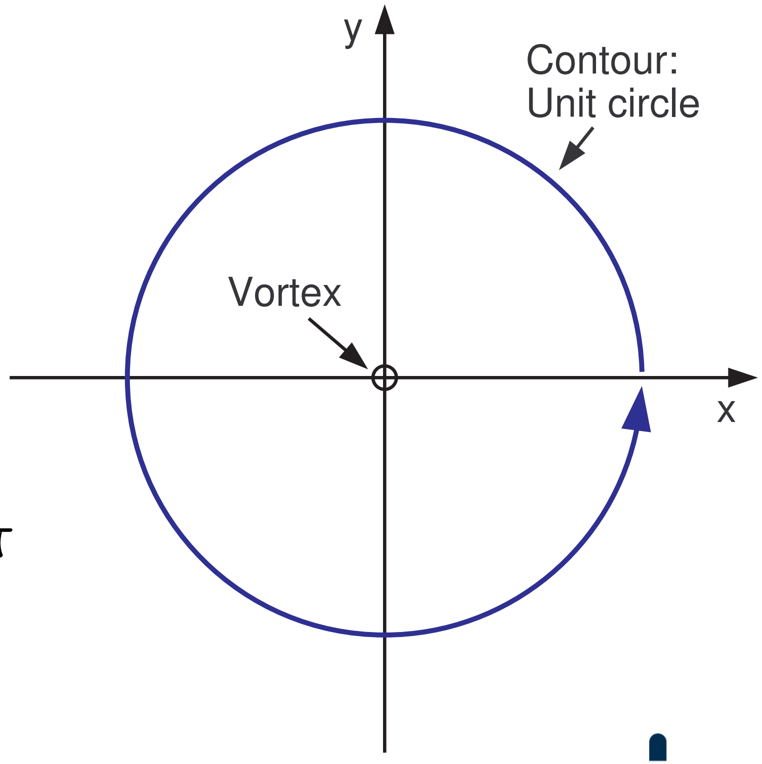


CSIR
our future through science

Topological charge

$$V_- = x - iy = \rho \exp(-i\phi)$$
$$V_+ = x + iy = \rho \exp(i\phi)$$



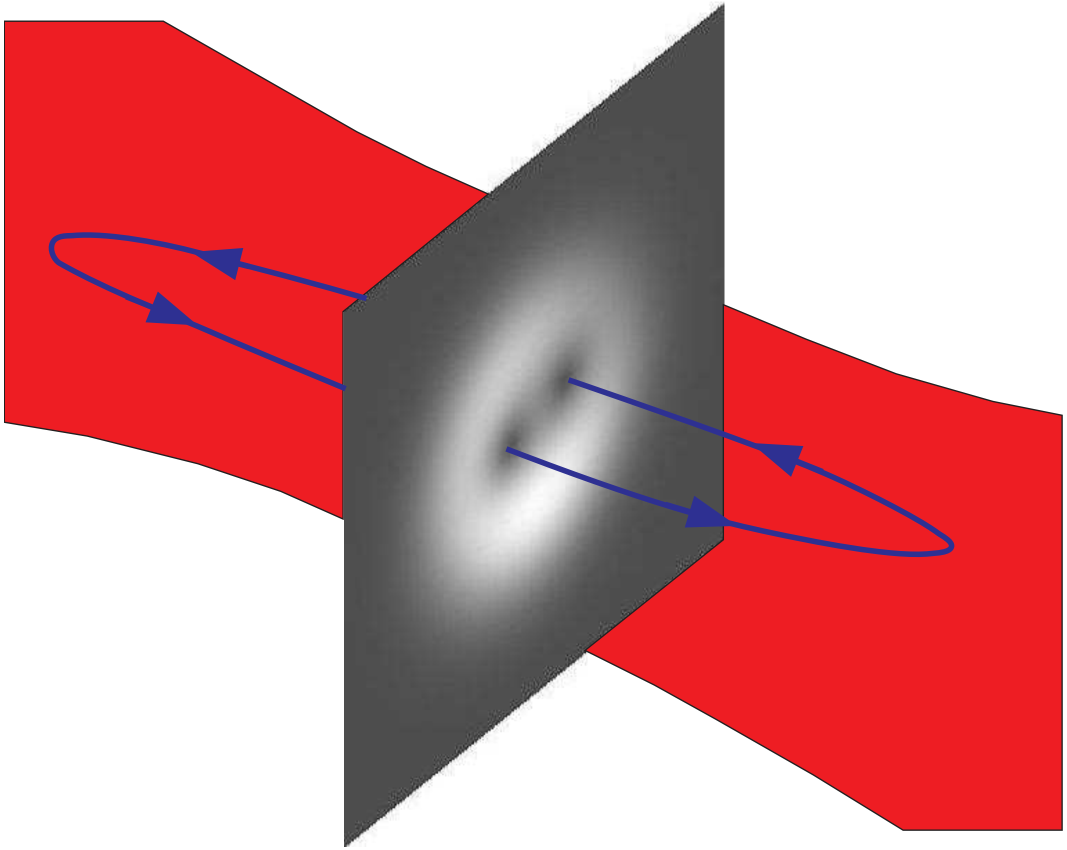


$$\oint_C \nabla\theta(x,y) \cdot \hat{\mathrm{d}}s = \nu \, 2\pi$$

Vortex dipole = 2 oppositely charged vortices

Topological charge conservation

Vortices form lines in 3D
$\longrightarrow$ annihilation and creation of vortex dipoles
$\longrightarrow$ topological charge flow



Topological charge is locally conserved.
Net flow of topological charge into a closed volume is zero.

CSIR
our future through science

Parameterisation

Taylor series expansion around vortex at origin

positive vortex

negative vortex

$$0 + a_x x + a_y y + \ldots = A \exp(i\Omega) \left[\xi (x + iy) + \zeta (x - iy) \right]$$

complex coefficients

morphology parameters

overall amplitude and phase

where $|\xi|^2 + |\zeta|^2 = 1$

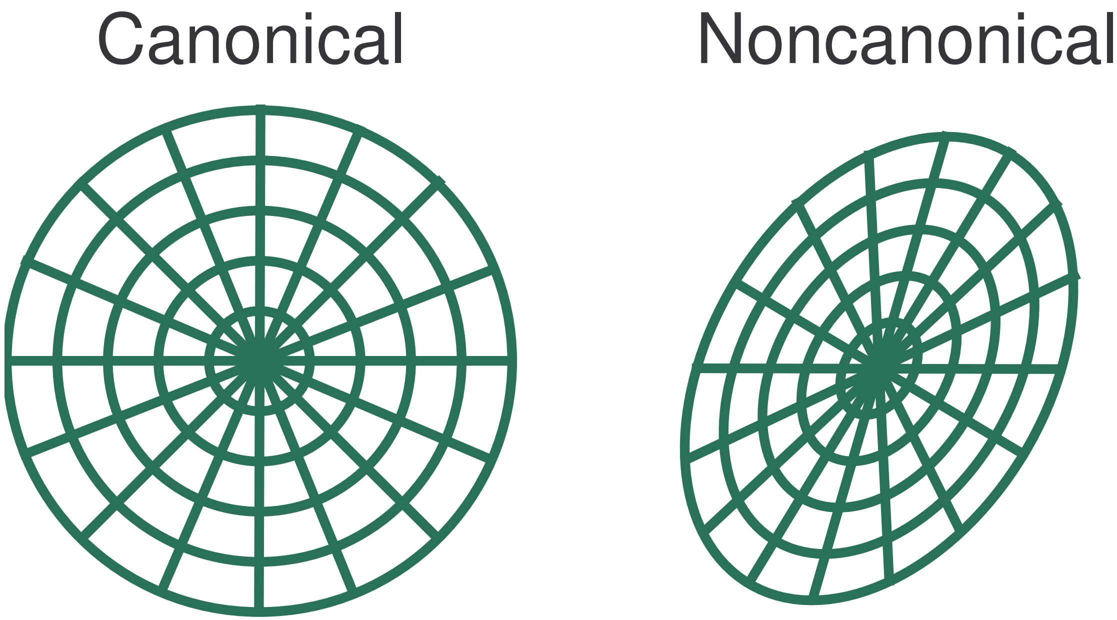
our future through science

Vortex shape

For isotropic (canonical) vortices:

$\xi = 1$ and $\zeta = 0 \quad \rightarrow \quad \nu = +1$
$\xi = 0$ and $\zeta = 1 \quad \rightarrow \quad \nu = -1$

All other cases are anisotropic (noncanonical)



Vortex morphology

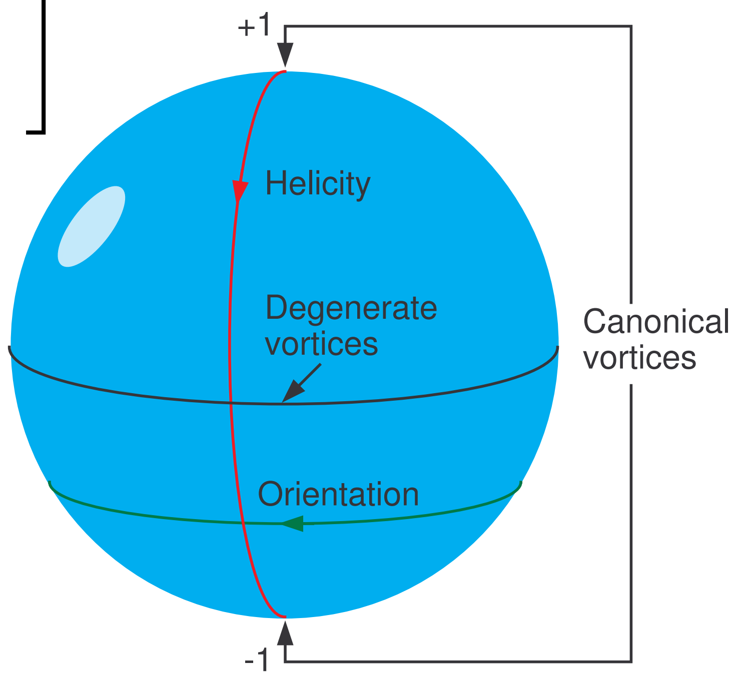
Morphology of the vortex (anisotropy and orientation) is given by ξ and ζ.
Analogues to Jones vectors (polarisation):

$$\eta = \begin{bmatrix} \xi \\ \zeta \end{bmatrix} = \begin{bmatrix} \cos(\psi/2)\exp(i\beta/2) \\ \sin(\psi/2)\exp(-i\beta/2) \end{bmatrix}$$

Morphology angles:

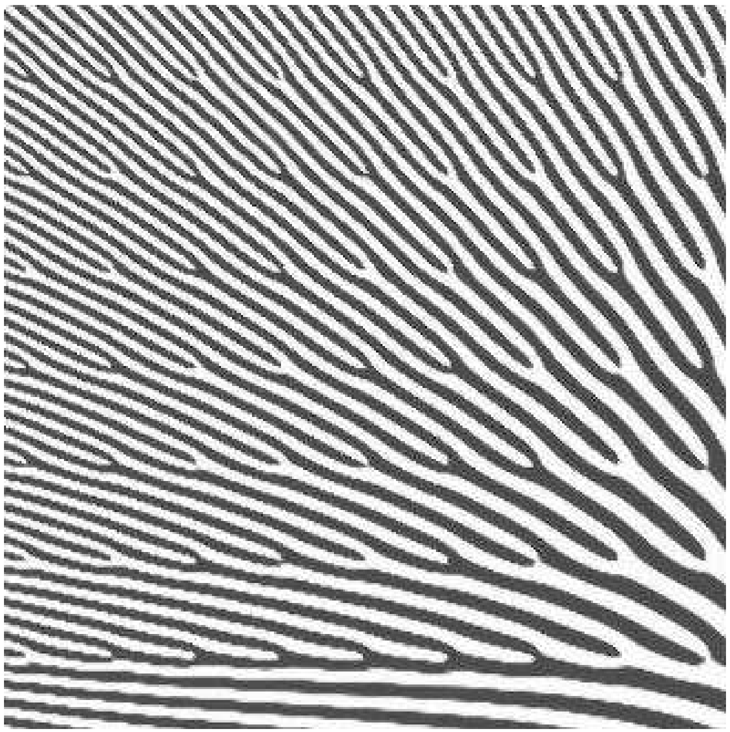
- ▷ $0 \leq \psi \leq \pi$ — helicity ($\cos\psi$)
- ▷ $0 \leq \beta < 2\pi$ — orientation

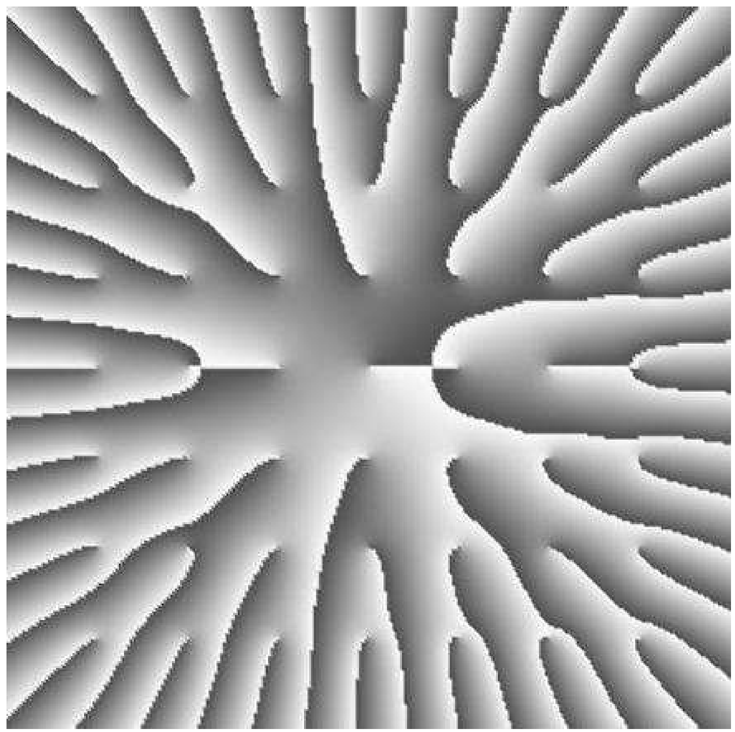
Coordinates on a Bloch sphere



Generating optical vortices

Diffractive optical elements (DOEs)
or Spatial light modulators (SLMs)
For phase function (with vortices): $\theta(x, y)$
Amplitude function: $t(x, y) = \frac{1}{2} + \frac{1}{2} \cos\left[\theta(x, y)\right]$
Phase functions: $t(x, y) = \exp\left[i\theta(x, y)\right]$



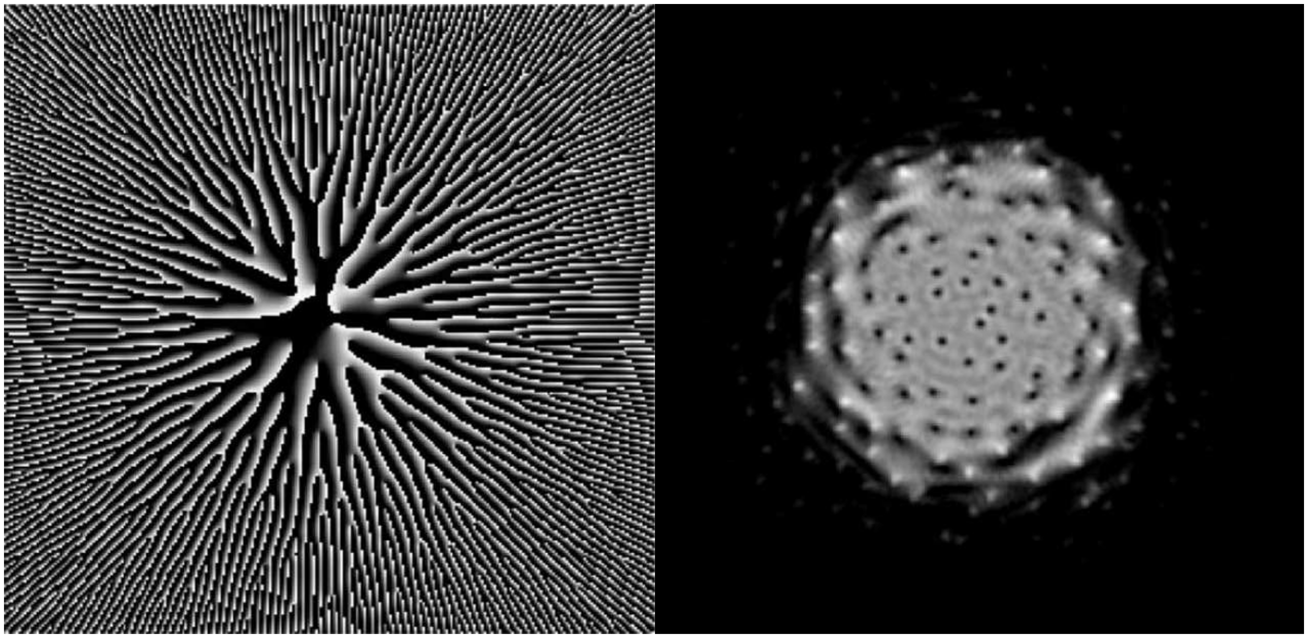


our future through science

Vortex density limitation

The maximum topological charge density D in area

with circumference L for wavelength λ: $$\frac{D}{L} < \frac{1}{\lambda}$$

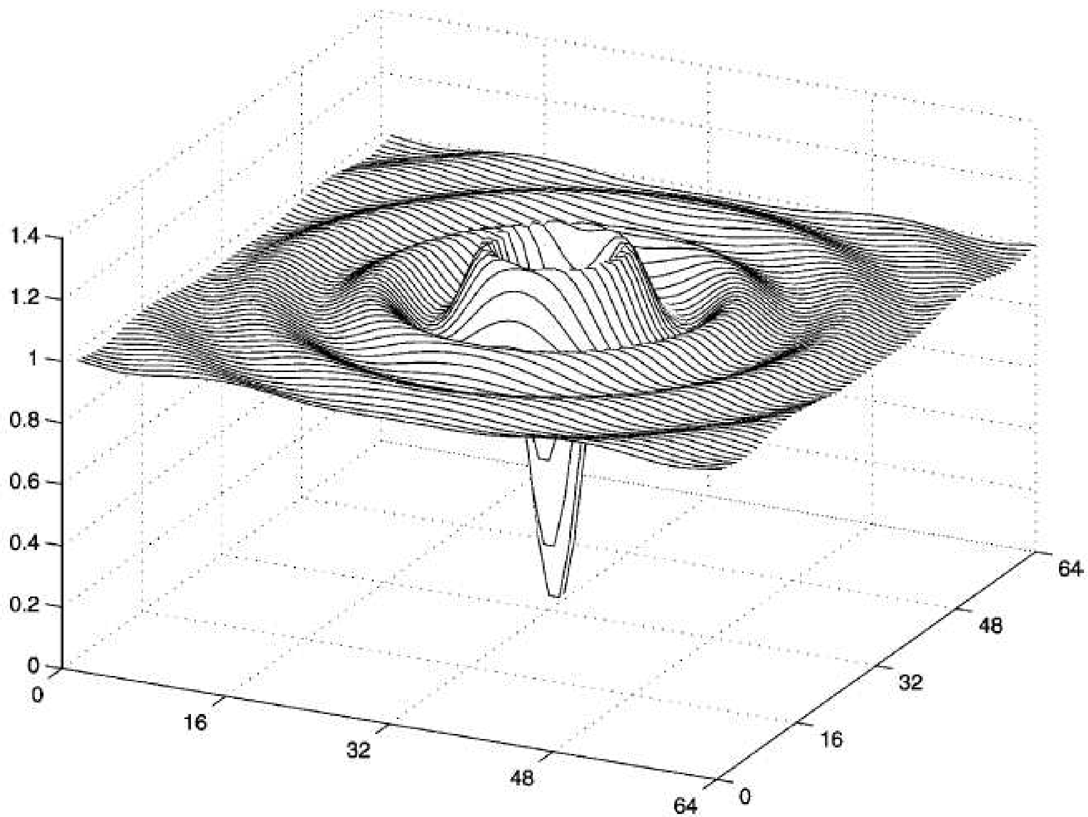


For spatial frequency $>$ wavenumber
$\Rightarrow$ purely evanescent waves
$\Rightarrow$ exterior depletion

CSIR
our future through science

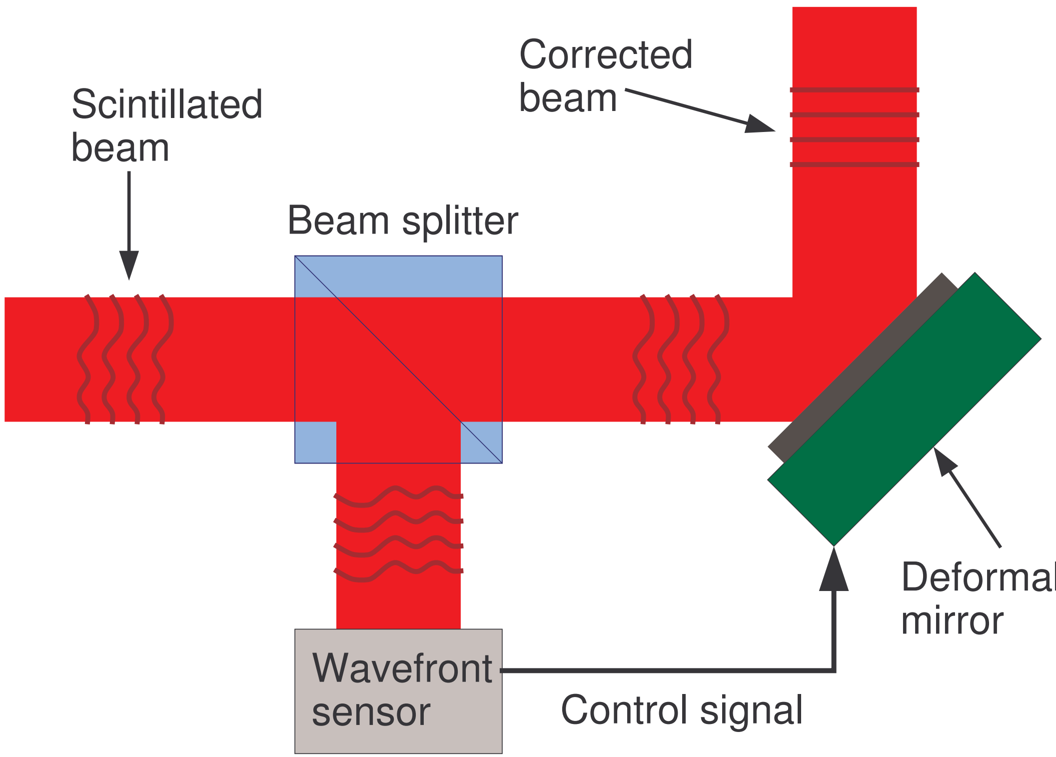
Point vortex profile

Phase only element → point vortices
(no amplitude modulation)
Density limitation → effective profile for point vortex
(remove evanescent field)



Scintillated optical beams

Optical beam in a turbulent atmosphere:
$\longrightarrow$ index variations cause random phase modulations
$\longrightarrow$ leads to distortion of the optical beam.
Weak scintillation $\longrightarrow$ continuous phase distortions that can be corrected by an adaptive optical system:

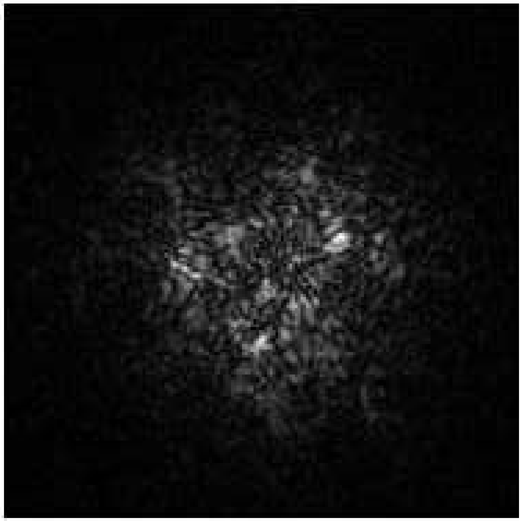


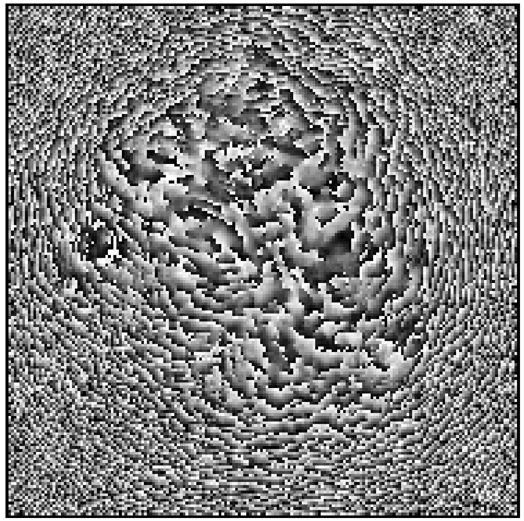
CSIR
our future through science

Strong scintillation

Strong scintillation $\rightarrow$ optical vortices.

$\Rightarrow$ conventional adaptive optics does not work anymore.



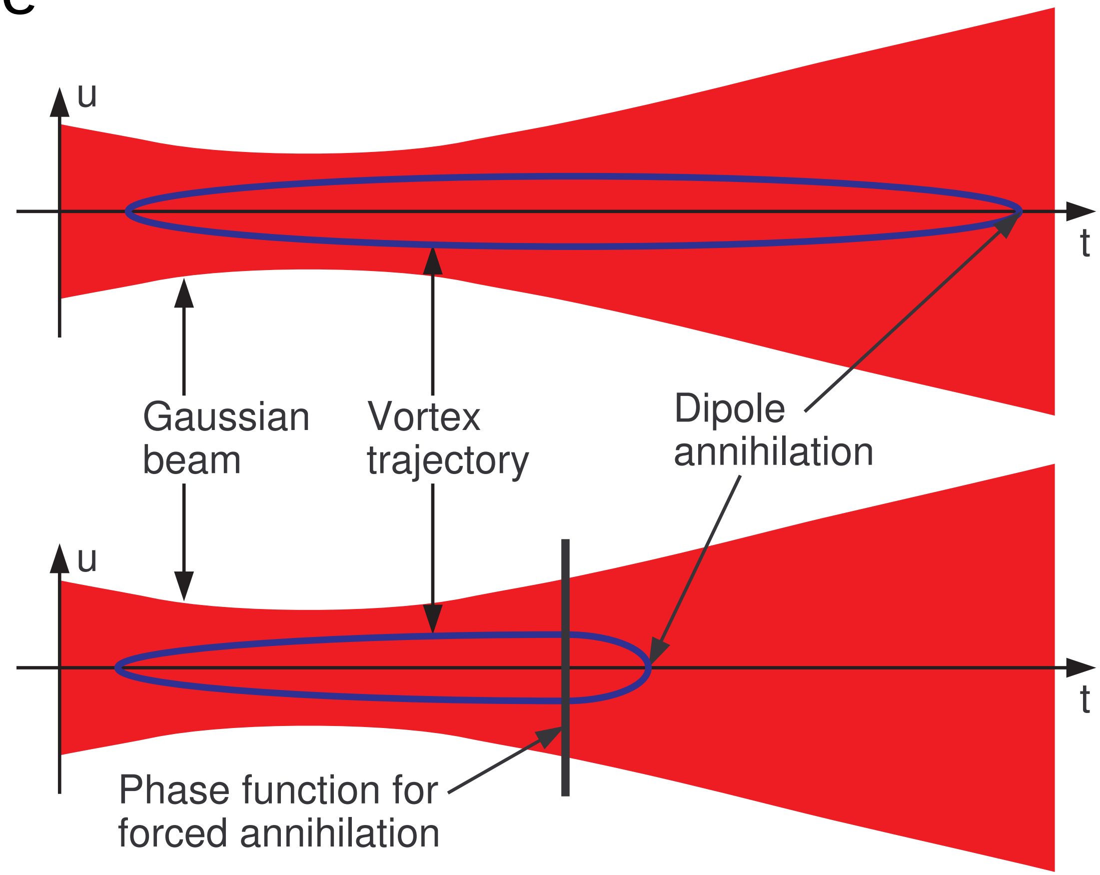


Need to get rid of the vortices.

our future through science

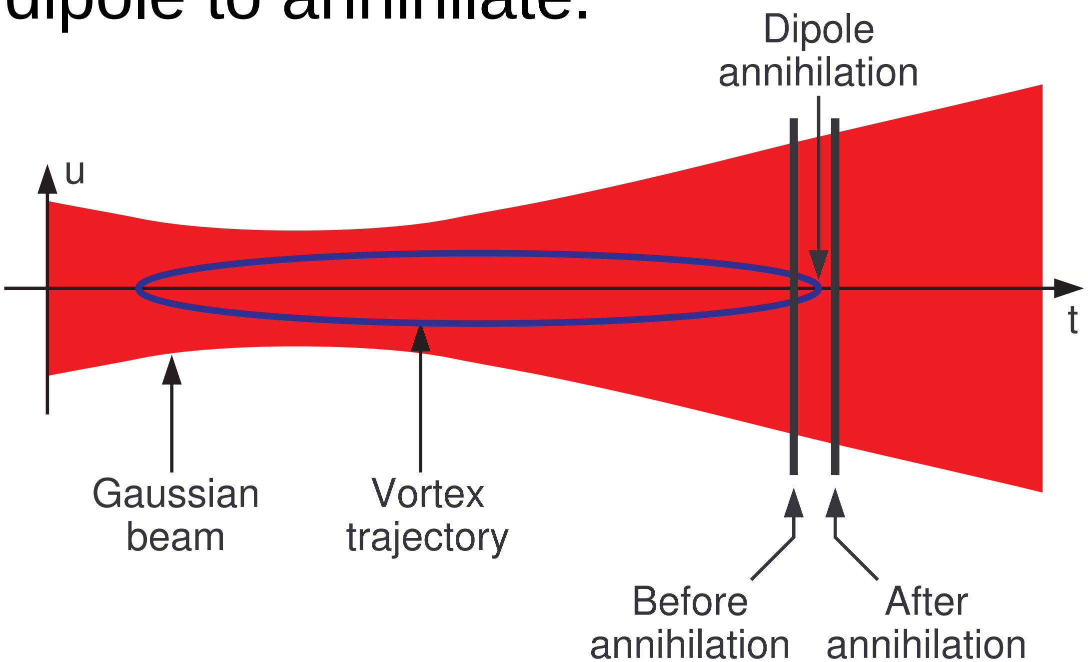
Forced annihilation

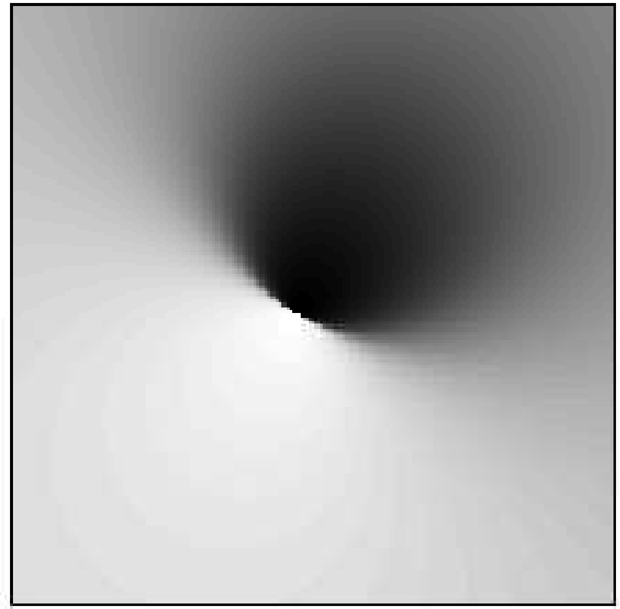
One idea to get rid of optical vortices in strongly scintillated optical beams is to force vortex dipoles to annihilate sooner by introducing a special phase function.



Annihilation in Gaussian beam

The phase function behind an annihilation point looks similar to the continuous part of the phase function before the annihilation point. The latter somehow has the ability to cause annihilation. So one can use the former to force a vortex dipole to annihilate.





CSIR
our future through science

Vortex removal procedure

Procedure to remove optical vortices:

- ▷ Located all the optical vortices.
- ▷ Divide them into dipoles.
- ▷ Compute annihilation phase function for each dipole.
- ▷ Multiply beam with all these phase functions.
- ▷ Allow beam to propagate.

Not extremely successfull!

We need to understand the statistical behaviour of vortex fields better.

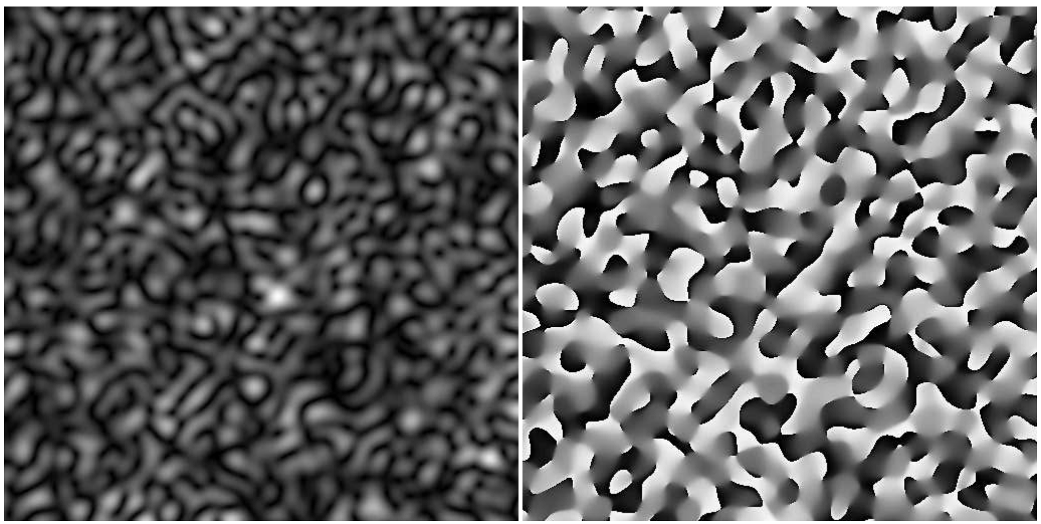
our future through science

Random vortex fields

A random wave field (speckle field) = random vortex fields.

$$f(x, y, z) = \sum_n \alpha_n \exp\left(i\mathbf{k}_n \cdot \mathbf{x}\right)$$

$\mathbf{k}_n$ — propagation vectors inside cone angle θ
α_n — random complex coefficients



Amplitude Phase

our future through science

Properties of random vortex fields

(... or what we already know)

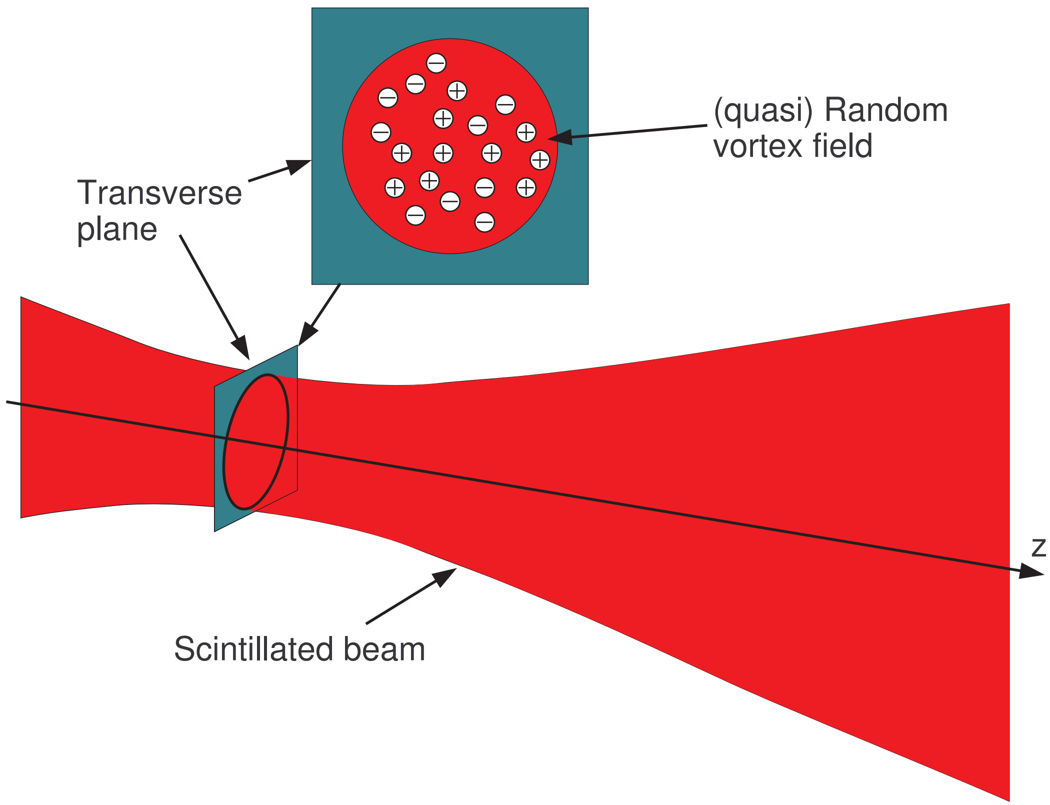
- ▷ Vortex density inversely proportional to coherence area.
- ▷ Globally $\Rightarrow$ neutral topological charge.
- ▷ Adjacent topological charges are anti-correlated.
- ▷ Annihilation rate = creation rate

$\Rightarrow$ System in equilibrium

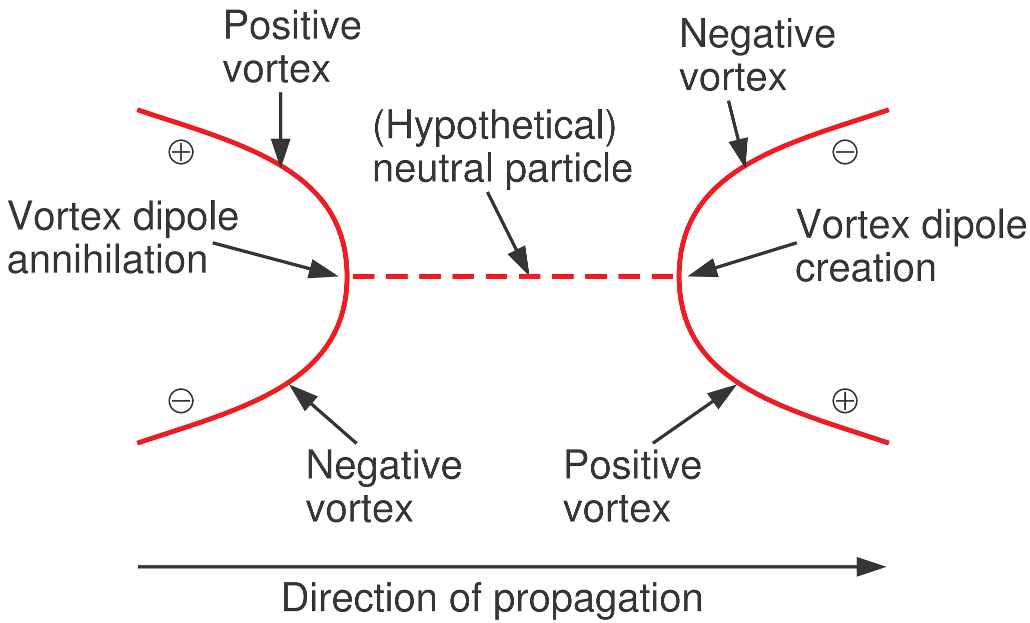
CSIR
our future through science

Perspective

Physical beam: 2+1 dimensional world.
Vortices = particles in 2D space.
Propagation direction = time-dimension.



Vortex plasma model



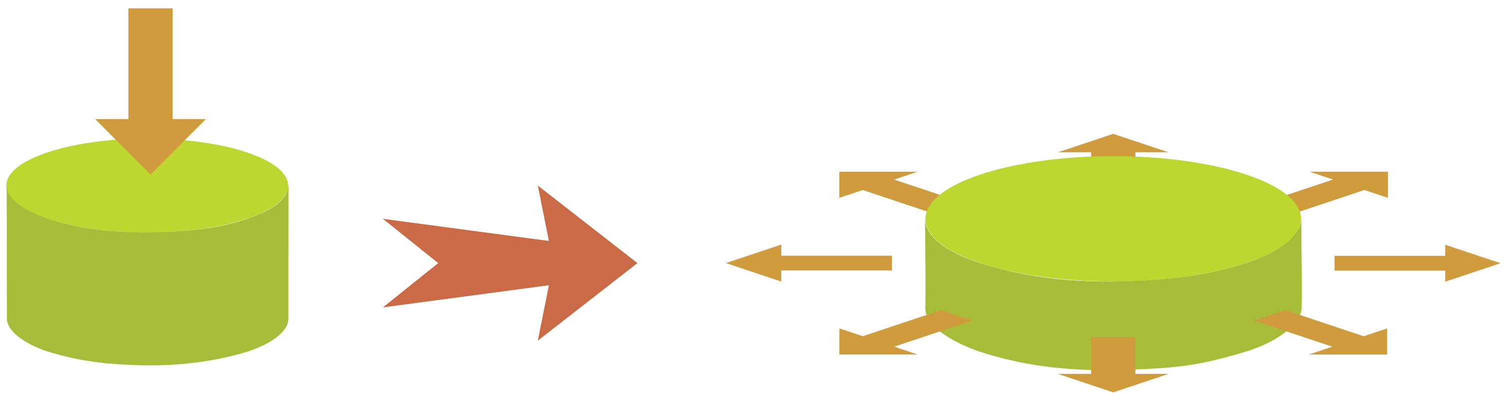
Three types of particles:

- ▷ Positive vortices $n_p(x, y, z)$
- ▷ Negative vortices $n_n(x, y, z)$
- ▷ Neutral bound states $n_0(x, y, z)$

  (positive vortex + negative vortex)

CSIR
our future through science

Conservation equations



$$
\begin{aligned}
(1) \quad & \partial_z n_p(x,y,z) + \nabla \cdot \mathbf{J}_p(x,y,z) &=& \ \mathcal{C} - \mathcal{A} \\
(2) \quad & \partial_z n_n(x,y,z) + \nabla \cdot \mathbf{J}_n(x,y,z) &=& \ \mathcal{C} - \mathcal{A} \\
(3) \quad & \partial_z n_0(x,y,z) + \nabla \cdot \mathbf{J}_0(x,y,z) &=& \ \mathcal{A} - \mathcal{C},
\end{aligned}
$$

$\mathbf{J}_p(x,y,z)$, $\mathbf{J}_n(x,y,z)$ and $\mathbf{J}_0(x,y,z)$ — currents associated with $n_p(x,y,z)$, $n_n(x,y,z)$ and $n_0(x,y,z)$, respectively.

$\mathcal{C}$ — rate of creations

$\mathcal{A}$ — rate of annihilations

our future through science

Vortex number conservation

Total positive/negative vortex number density:

$$Q_P = n_p + n_0 \qquad\qquad Q_N = n_n + n_0$$

Total positive/negative vortex number conservation:

$$\partial_z Q_P + \nabla \cdot \mathbf{J}_P = 0 \qquad \text{where} \qquad \mathbf{J}_P = \mathbf{J}_p + \mathbf{J}_0$$

$$\partial_z Q_N + \nabla \cdot \mathbf{J}_N = 0 \qquad \text{where} \qquad \mathbf{J}_N = \mathbf{J}_n + \mathbf{J}_0$$

Total positive vortex number density:

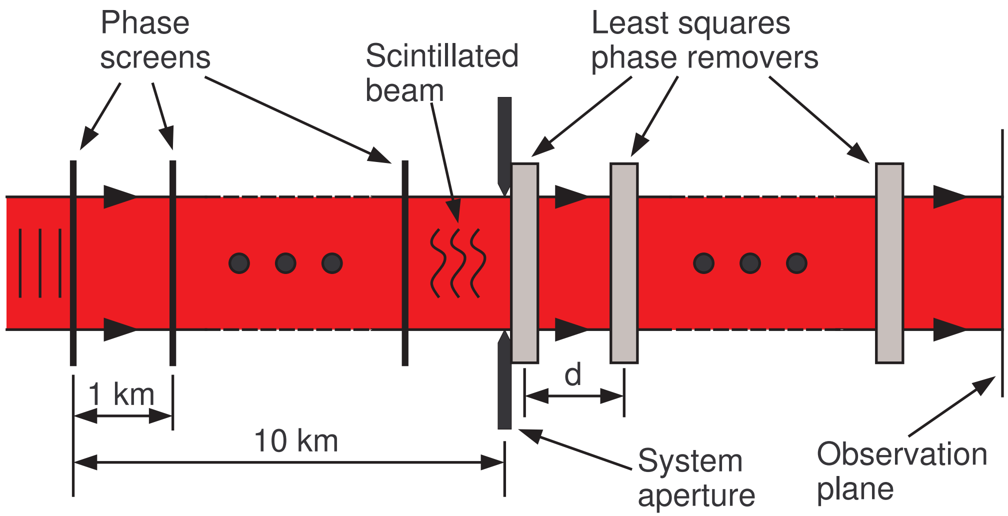
$$V = Q_P + Q_N = n_p + n_n + 2n_0$$

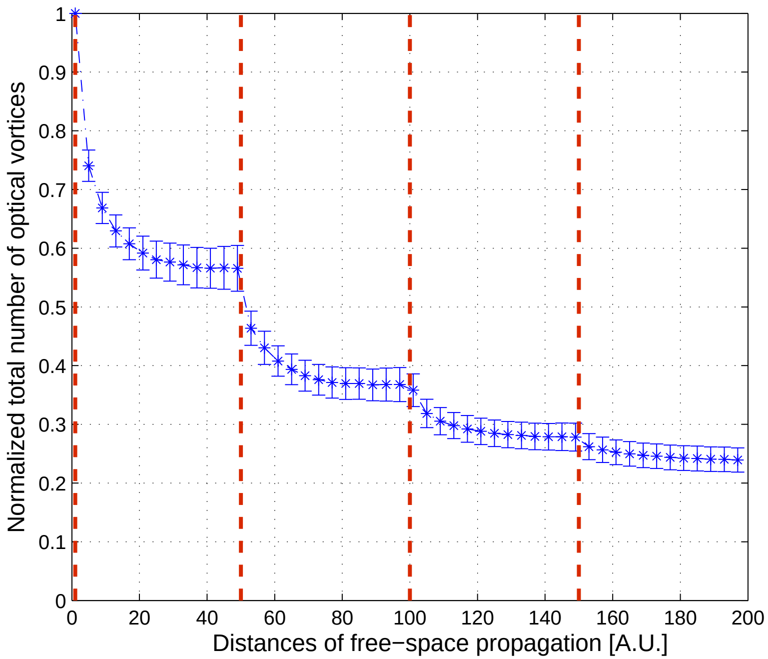
Total vortex number conservation:

$$\partial_z V + \nabla \cdot \mathbf{J}_V = 0 \qquad \text{where} \qquad \mathbf{J}_V = \mathbf{J}_P + \mathbf{J}_N$$

Disturbing the equilibrium

Removing the continuous phase repeatedly





Initial number of optical vortices is reduced asymptotically until equilibrium is reached.

our future through science

Homogeneous case

Remove continuous phase $\rightarrow$ reduce $n_0(z)$
$\Rightarrow$ not in equilibrium
Homogeneous + density limitation:
$\Rightarrow$ globally neutral [$n_p(z) = n_n(z)$]
Assume that:

- $\triangleright$ Rate of creations is proportional to n_0 (!)
- $\triangleright$ Rate of annihilations is proportional to n_p

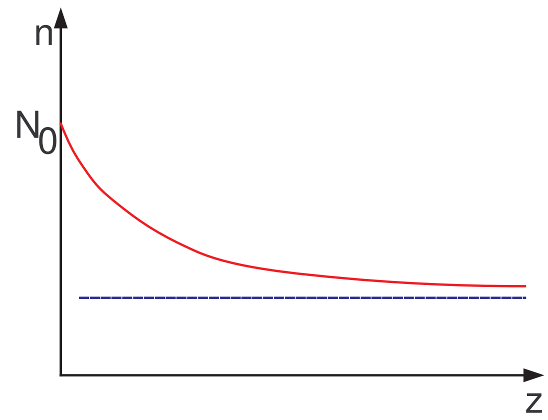
For homogeneous densities: $\nabla \cdot \mathbf{J} = 0$
$\Rightarrow$ Conservation equations become rate equation:

$$\partial_z n_p(z) = \frac{n_0(z)}{\tau_0} - \frac{n_p(z)}{\tau_p}$$

$$\partial_z n_0(z) = \frac{n_p(z)}{\tau_p} - \frac{n_0(z)}{\tau_0}$$

CSIR our future through science

Solution



Conservation of positive vortex number:

$\Rightarrow n_p(z) + n_0(z) = Q_P(z) = N_0$ (constant)

Solution:

$$n_p(z) = \frac{N_0 \tau_p}{\tau_0 + \tau_p} + \left(n_p(0) - \frac{N_0 \tau_p}{\tau_0 + \tau_p} \right) \exp \left[-\frac{(\tau_0 + \tau_p) z}{\tau_0 \tau_p} \right]$$

If we assume: $n_0(0) = 0 [\Rightarrow n_p(0) = N_0]$

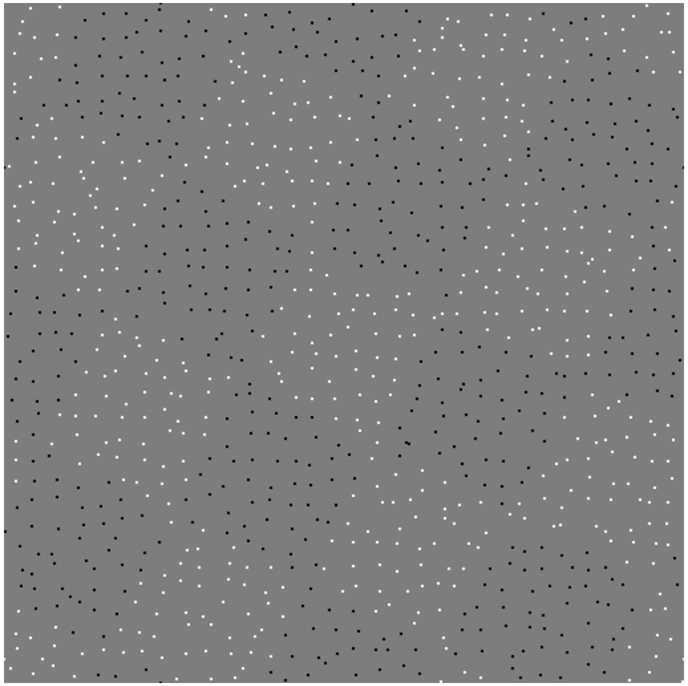
$$n_p(z) = \frac{N_0 \tau_p}{\tau_0 + \tau_p} + \frac{N_0 \tau_0}{\tau_0 + \tau_p} \exp \left[-\frac{(\tau_0 + \tau_p) z}{\tau_0 \tau_p} \right]$$

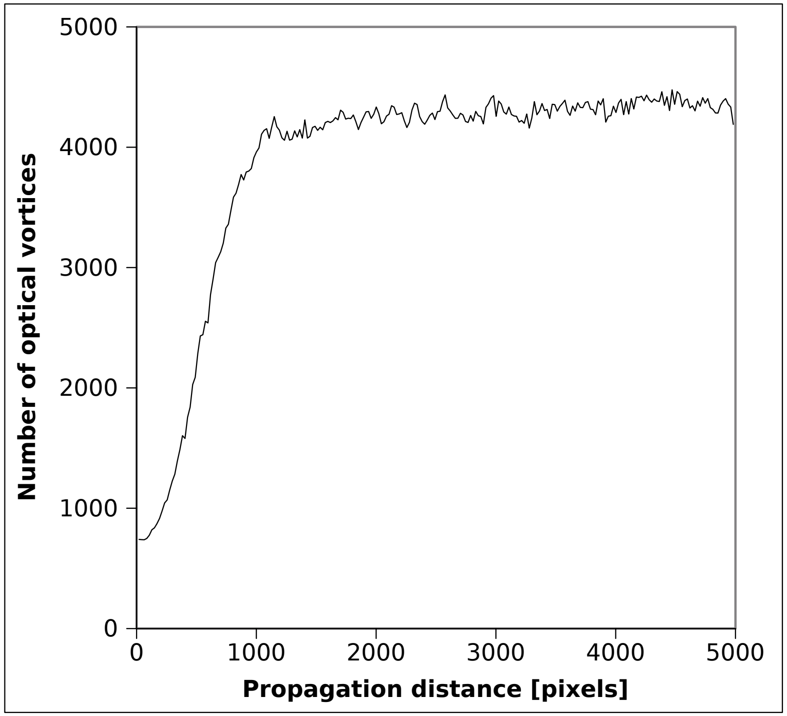
Solution qualitatively agrees with observation, but since $n_0(0)$ is not known, one cannot solve for τ_p and τ_0.

CSIR our future through science

Inhomogeneous case

Numerical simulations for densities having regions with different topological charges.



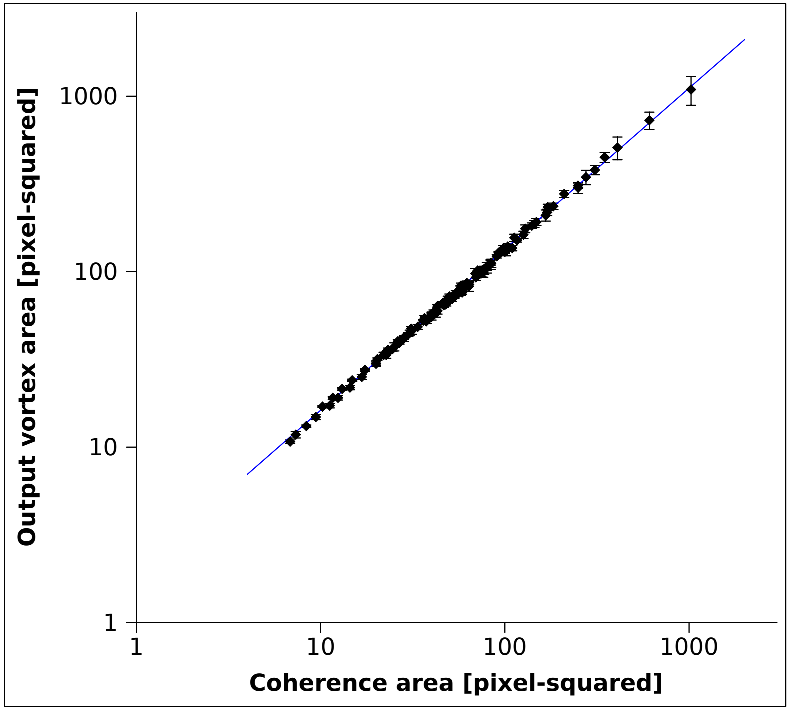


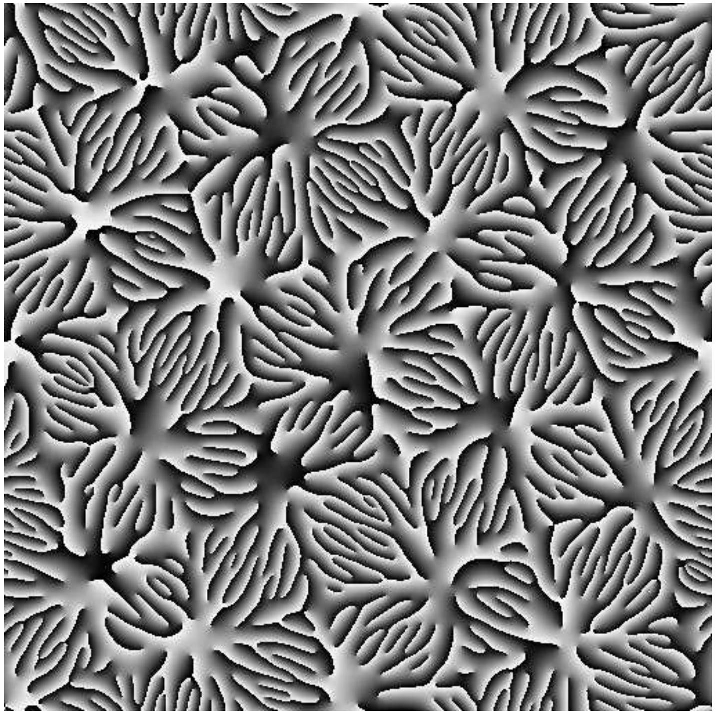
Vortex number increases during propagation.
Restoration scale proportional to A_{in}/λ

our future through science

Why larger vortex density?

Coherence area $\rightarrow$ autocorrelation peak.





Separated topological charge:

$\Rightarrow$ higher spatial frequencies

$\Rightarrow$ narrower autocorrelation peak

$\Rightarrow$ smaller coherence area

$\Rightarrow$ larger vortex density

CSIR our future through science

Topological charge density

Total vortex number depends on the hypothetical existence of n_0.

To avoid n_0 consider the topological charge density:

$$D = n_p - n_n \qquad \mathbf{J}_D = \mathbf{J}_p - \mathbf{J}_n$$

Conservation of topological charge density:

$$\partial_z D(x, y, z) + \nabla \cdot \mathbf{J}_D(x, y, z) = 0$$

our future through science

Diffusion equation

Random motion of vortices $\rightarrow$ random walk
$\Rightarrow$ Diffusion equation:

$$\partial_z D(\mathbf{x}, z) - \kappa \nabla^2 D(\mathbf{x}, z) = 0$$

where κ is the diffusion coefficient and $D(\mathbf{x}, z)$ is the topological charge density (TCD)

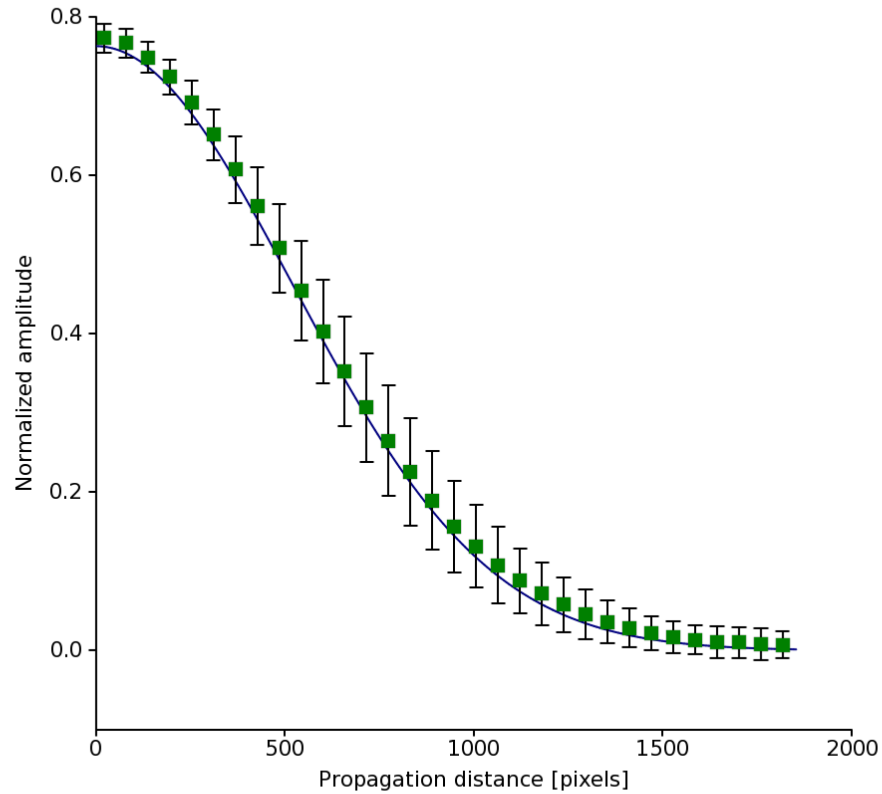
Solutions (exponential decay):

$$D(\mathbf{x}, z) = \exp(-\kappa |\mathbf{a}|^2 z) \cos(\mathbf{a} \cdot \mathbf{x})$$

Rate of decay increases for higher spatial frequencies.

Numerical simulations (1D)

Numerical simulation:
Propagate initial beam cross-section:
$D(\mathbf{x}, z = 0) = \cos(ax)$
$\longrightarrow$ compute TCDs
$\longrightarrow$ extract FT component



Found Gaussian shape!

$\Rightarrow$ broken shift invariance

Modified equation: $\partial_z D - \kappa_0 z \nabla^2 D = 0$

General solution: $$\exp\left(-\frac{\kappa_0}{2}|\mathbf{a}|^2 z^2\right)\cos(\mathbf{a}\cdot\mathbf{x})$$

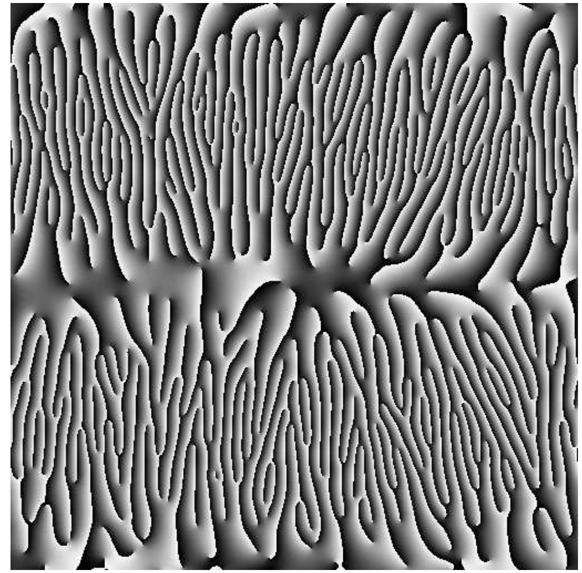
CSIR our future through science

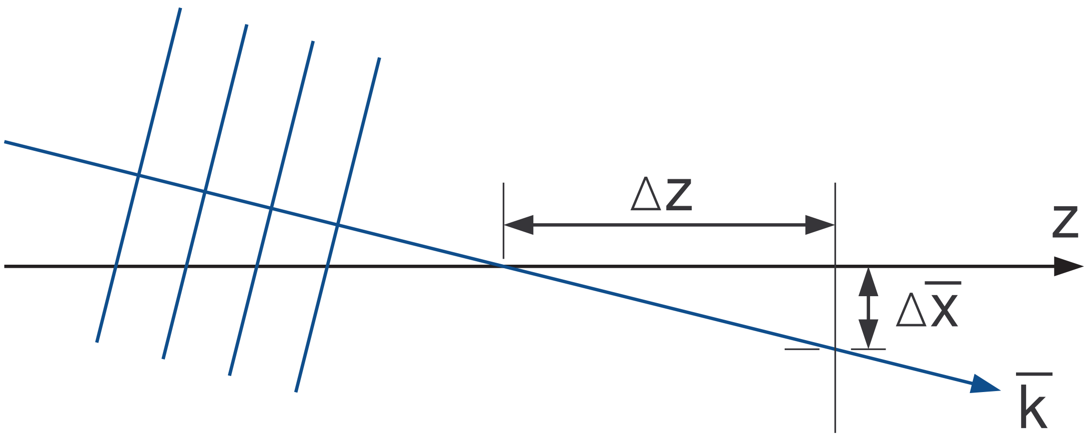
Phase drift

Concentrated topological charge

$\rightarrow$ phase slope $\rightarrow$ sideways drift

$$\Delta \mathbf{x} \approx \frac{-\nabla\theta\Delta z}{k} \quad \text{for} \quad k \gg |\nabla\theta|$$





$$D(\mathbf{x}, z + \Delta z) = D(\mathbf{x} - \Delta \mathbf{x}, z) \approx D(\mathbf{x}, z) - \Delta \mathbf{x} \cdot \nabla D(\mathbf{x}, z)$$

for $\Delta z \rightarrow 0$: $$\partial_z D(\mathbf{x}, z) = \frac{\nabla\theta \cdot \nabla D(\mathbf{x}, z)}{k}$$

Drift term

Gradient of the phase function: $\nabla\theta = D(\mathbf{x}, z) \otimes \nabla\phi$

Drift term:

$$\partial_z D(\mathbf{x}, z) = \frac{1}{k}\left[D(\mathbf{x}, z) \otimes \nabla\phi\right] \cdot \nabla D(\mathbf{x}, z)$$

where $\otimes$ denotes convolution; and

$\nabla\phi$ is the phase gradient of a single vortex:

$$\nabla\phi(x, y) = \frac{y\hat{x} - x\hat{y}}{x^2 + y^2}$$

CSIR

our future through science

Full Fokker-Planck equation

$$\partial_z D - \kappa_0 z \nabla^2 D - \frac{1}{k} (D \otimes \nabla \phi) \cdot \nabla D = 0$$

D — topological charge density

κ_0 — dimensionless diffusion parameter

k — wavenumber

ϕ — phase function of a single canonical vortex

Two-dimensional example

Initial distribution:

$$D(x, y, z) = \cos(ax) + \cos(by) \qquad \nabla\phi(x, y) = \frac{y\hat{x} - x\hat{y}}{x^2 + y^2}$$

Convolution:

$$D \otimes \nabla\phi = \frac{-2\pi\hat{y}}{a}\sin(ax) + \frac{2\pi\hat{x}}{b}\sin(by)$$

Gradient: $\nabla D = -a\hat{x}\sin(ax) - b\hat{y}\sin(by)$

Drift term:

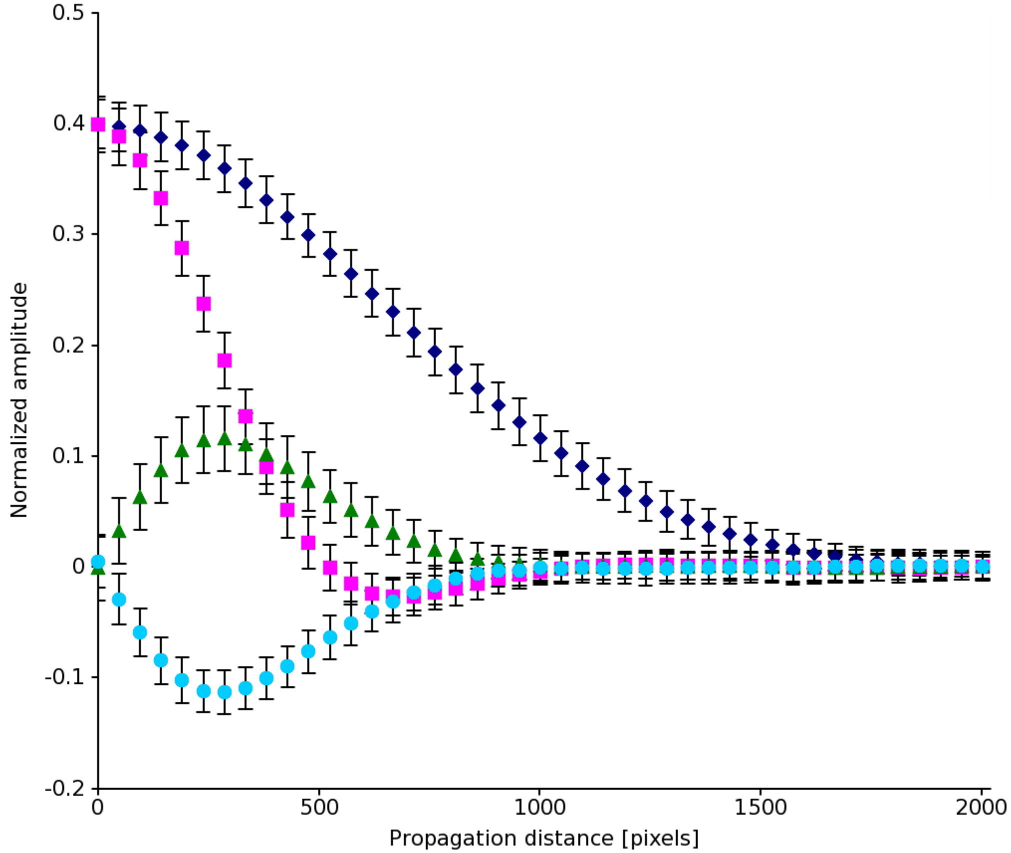
$$(D \otimes \nabla\phi) \cdot \nabla D = 2\pi\frac{b^2 - a^2}{ab}\sin(ax)\sin(by)$$

$$(D \otimes \nabla\phi) \cdot \nabla D = 2\pi\frac{b^2 - a^2}{ab}\left[\cos(ax - by) - \cos(ax + by)\right]$$

CSIR
our future through science

Nonlinearity

The drift term only contributes for Fourier components with different magnitudes and different directions: $\cos(ax)$ and $\cos(by)$



New Fourier components:
$\cos(ax + by)$ and $\cos(ax - by)$
are generated due to the nonlinear drift term.

Conclusions

- Strongly scintillated beams contain vortex fields that need to be removed to correct the beam.
- Special phase functions can force isolated dipoles to annihilate, but doesn’t work well for vortex fields.
- The number of vortices are reduced by removing the continuous phase from homogeneous vortex fields.
- Vortex number increase during propagation of inhomogeneous vortex fields.
- A vortex plasma model is used to model the behaviour of vortex fields.

our future through science

More conclusions

- Vortex fields can be described by a Fokker-Planck equation that contains a drift term and a diffusion term.
- For one-dimensional distributions the drift term falls away and the result is a heat equation, but the decay follows a Gaussian shape because the system lacks shift invariance.
- For some two-dimensional distributions the drift term is nonzero and generates new Fourier components due to its nonlinear nature.

Mathematical Optics Team

our future through science