General aspects of optical vortices

F. Stef Roux
CSIR National Laser Centre
PO Box 395, Pretoria 0001, South Africa
Contents

- Definition of an optical vortex
- Topological charge and vortex morphology
- How to detect a vortex — interferometry
- How to generate optical vortices
Persistent dark spots

Optical vortices
Speckle field

Amplitude

Phase
Singular phase function
Null crossings

Real part

Optical vortex

Null crossing

Imaginary part

Null crossing
Topological charge

-1

+1
Expression of a canonical vortex

Basic complex valued function for a vortex (at origin):

Positive: $V_+(x, y) = x + iy = \rho \exp(i\phi)$

Negative: $V_-(x, y) = x - iy = \rho \exp(-i\phi)$
Topological charge integral

Index integral:
Integrate gradient of the phase function around a closed contour that encloses the phase singularity.

\[ \oint_C \nabla \theta(x, y) \cdot \hat{d}s = \nu \, 2\pi \]
Topological charge computation

\[ \theta(x, y) = -i \ln \left( \frac{f(x, y)}{|f(x, y)|} \right) \Rightarrow \theta_\rho(\rho, \phi) = \phi \]

\[ \nabla \theta = \nabla \phi = \frac{1}{\rho} \hat{\phi} \quad \text{and} \quad \hat{d}s = \rho \hat{\phi} d\phi \]

\[ \oint \nabla \theta \cdot \hat{d}s = \int_0^{2\pi} d\phi = 2\pi \]

\[ \Rightarrow \text{Topological charge: } \nu = +1 \]
Profile function

Linear profile function: Profile function after CGH:

Amplitude

Distance
Laguerre-Gaussian beams

\[ \text{LG}_{nm}(r, \phi, t) = K \exp(i m \phi) L^{|m|}_n \left( \frac{2r^2}{1 + t^2} \right) \exp \left( \frac{-r^2}{1 - it} \right) \]

Normalised coordinates:

\[(u, v, t) = \left( \frac{x}{\omega_0}, \frac{y}{\omega_0}, \frac{z}{\rho} = \frac{z\lambda}{\pi\omega_0^2} \right) \]

\(L^{|m|}_n\) — ass. Laguerre polynomials; \(n\) — nonzero integer \(m\) — signed integer (Topological charge) \(K\) — normalisation constant:

\[K = \frac{r^{|m|}(1 + it)^n}{(1 - it)^{n+|m|+1}} \sqrt{\frac{n!2^m+1}{\pi(n + m)!}}\]
For $m \neq 0$ the Laguerre-Gaussian beam has an optical vortex on the axis with topological charge $m$.

The Laguerre-Gaussian beams carry orbital angular momentum (OAM), which can be transferred to small particles.

The amount of OAM is proportional to the intensity and proportional to $m$. 
Higher order topological charge

One can have vortices with arbitrary high topological charge, but they are not stable.
General complex expression

Taylor series expansion of arbitrary complex valued function around a (1st order) vortex at \((x_0, y_0)\):

\[ f(x, y) = 0 + a_x(x - x_0) + a_y(y - y_0) + \ldots \]

Complex valued coefficients:

\[ a_x = a_{xr} + i a_{xi} \quad a_y = a_{yr} + i a_{yi} \]

\( \Rightarrow \) 6 real valued parameters for each vortex:

\[ a_{xr}, a_{xi}, a_{yr}, a_{yi}, x_0, y_0 \]
Parameterisation

Remove global amplitude and global phase:

\[ a_xx + a_y y = A \exp(i\Omega) [\xi(x + iy) + \zeta(x - iy)] \]

where \(|\xi|^2 + |\zeta|^2 = 1\)

For canonical vortices:
\( \xi = 1 \) and \( \zeta = 0 \) (\( \nu = +1 \)) or \( \xi = 0 \) and \( \zeta = 1 \) (\( \nu = -1 \))

Other cases: noncanonical
Vortex morphology

Shape (morphology) of the vortex is given by $\xi$ and $\zeta$. Analogues to Jones vectors (polarisation):

$$\eta = \begin{bmatrix} \xi \\ \zeta \end{bmatrix} = \begin{bmatrix} \cos(\psi/2) \exp(i\beta/2) \\ \sin(\psi/2) \exp(-i\beta/2) \end{bmatrix}$$

Morphology angles:
- $0 \leq \psi \leq \pi$ — helicity ($\cos \psi$)
- $0 \leq \beta < 2\pi$ — orientation
Noncanonical vortex notation

Shifted noncanonical vortex in helical coordinates:

\[ V = \xi (w - w_0) + \zeta (\overline{w} - \overline{w}_0) \]

Helical coordinates: \( w = u + iv \) and \( \overline{w} = u - iv \)

Vortex location in terms of helical coordinates:
\( w_0 = u_0 + iv_0 \) and \( \overline{w}_0 = u_0 - iv_0 \)
Noncanonical vortex example
The morphology angle can be seen as angular coordinates for the surface of a sphere, like the Poincaré sphere for polarisation.
Extracting morphology

Given an arbitrary complex function \( f(x, y) \) one can extract the morphology of any vortex. Define vortex derivatives:

\[
\partial_{\pm} f = \frac{1}{2} \left( \frac{\partial f}{\partial x} \pm i \frac{\partial f}{\partial y} \right)
\]

\[
\partial_+ V_+ = \partial_- V_- = 1 \quad \partial_+ V_- = \partial_- V_+ = 0
\]

\[
\cos \psi = \frac{|\partial_+ f|^2 - |\partial_- f|^2}{|\partial_+ f|^2 + |\partial_- f|^2} \quad \text{(helicity)}
\]

\[
\exp(i\beta) = \frac{\partial_+ f (\partial_- f)^*}{|\partial_+ f||\partial_- f|} \quad \text{(orientation)}
\]
Morphology distribution

Beam amplitude

Helicity
\[ |f + g|^2 = (f + g)(f^* + g^*) = |f|^2 + |g|^2 + 2|f||g| \cos (\theta_f - \theta_g) \]
Interference between a plane wave and a vortex beam gives a line with a branch point.
Holography

Transmission function:

\[ t(x, y) = t_0 + K \left( |f|^2 + |g|^2 + fg^* + f^*g \right) \]
$g_t(x, y) = g(t_0 + K|f|^2 + K|g|^2) + Kf|g|^2 + Kf^*g^2$
Computer generated hologram

Compute an artificial amplitude transmission function for an arbitrary phase function:

\[ t(x, y) = \frac{1}{2} + \frac{1}{2} \cos [\theta(x, y)] \]

\( \theta(x, y) \) — phase function to be implemented

For amplitude transmission function: \( 0 < |t(x, y)| < 1 \)

Phase transmission functions directly modulate the phase of the beam: \( \Rightarrow t(x, y) = \exp [i\theta(x, y)] \)
Generating optical vortices

One can thus produce computer generated transmission functions that will produce arbitrary numbers of vortices at specific locations.
Doughnut lens
Higher diffraction orders

Computer generated hologram with first order vortex
Spatial light modulators

Laser beam without vortex

Spatial light modulator

Phase function with phase singularity

Laser beam with vortex
Conclusions

- Optical vortices are phase singularities with integer topological charges.
- Anisotropy and orientation is given by the morphology parameters.
- One can detect optical vortices using interference.
- One can generate optical vortices using computer generated holograms.