Stochastic singular optics

Filippus S. Roux

CSIR National Laser Centre, P.O. Box 395, Pretoria 0001, South Africa

ABSTRACT

The study of optical vortices in stochastic optical fields involves various quantities, including the vortex density and topological charge density, that are defined in terms of local expectation values of distributions of optical vortices. For stochastic optical fields that are inhomogeneous or not normally distributed, these local quantities often have nontrivial transient evolution as a function of propagation distance. The field of stochastic singular optics strive, among other things, to understand this dynamics. Here we review the tools and challenges of stochastic singular optics and provide some details of recent progress in this field.

Keywords: Singular optics, stochastic optical field, optical vortex density, topological charge density

1. INTRODUCTION

Speckle patterns are a typical phenomenon in random optical fields, resulting from coherent light being scattered from a random rough surface. It has been studied extensively[1,2] and much has been written about it, includes the properties of the phase singularities,[3] also called optical vortices,[4] that appear in such random optical fields. Optical vortices and their related quantities received much attention within the context of deterministic optical beams,[5–7] and some work has also been done on the statistical properties of optical vortices in random optical fields.[3,8–15] The study of optical vortices and other singularities in optical fields is known as singular optics.[6]

The study of random optical fields includes aspects related to the phase of the random field, such as the distribution of phase singularities or the expectation value of the phase gradient. Optical vortices are ubiquitous in random optical fields.[8] The random nature of such fields, together with their coherence properties, implies that there are many points where destructive interference creates complex zeros. In a random optical field such vortices are evenly distributed and their topological charges are mixed such that neighbouring vortices tend to have opposite topological charge.[11] As a result, the topological charge density of a speckle field is on average zero.[14,16] On the other hand, the vortex density is not zero, it is given by the second derivative of the peak of the autocorrelation function of the speckle.[3]

These observations are true for fully developed speckle. More recently, these studies have been extended to include optical fields that have statistical properties that evolve during propagation.[17–19] For example, it is possible to produce stochastic optical fields where the topological charge density has some inhomogeneous spatial distribution. Although this distribution eventually decays to zero it displays some interesting transient behaviour. In a certain sense one can view the process whereby random phase modulations turn an optical field into fully developed speckle as an example of such a case.[20] The same can happen in cases where the field already looks like fully developed speckle. We'll call all such optical fields *stochastic optical fields*, including the random optical fields, which have stationary statistical properties.

We'll refer to the study of the statistical properties of optical vortices in stochastic optical fields as *stochastic singular optics*. This term may sound to some extent like an oxymoron: while singular optics deals with optical vortices, among other things, such singularities tend to be ill-defined in partially coherent fields, which one typically deals with in stochastic optics. However, stochastic singular optics is not interested in individual optical vortices, but with various quantities that are, in one way or another, related to optical vortices. These quantities include the various vortex distributions, which are discuss next.

Email: fsroux@csir.co.za

1.1 Vortex distributions in stochastic optical fields

One way to model a stochastic optical field is as an ensemble of fully coherent random optical fields that share some properties. For each element in the ensemble — each coherent optical field — one can find the locations of all phase singularities. The statistical vortex distributions are obtained as the ensemble averages of all the vortex distributions of the elements of the ensemble. One can also determine the statistical distribution of the vorticity,[15, 21] the phase gradient or any other such quantity in a similar way by computing these quantities for each element in the ensemble and then average these quantities over the ensemble.

The positive (negative) vortex density is defined as the expectation value for the number of positive (negative) vortices per unit area on a transverse plane of the optical field. We denote the positive and negative vortex densities by two-dimensional functions $n_{\mathrm{p}}(x, y)$ and $n_{\mathrm{n}}(x, y)$, respectively, where x and y are the coordinates on the transverse plane. When we also consider how these functions evolve as a function of propagation distance z, they become three-dimensional functions of $\{x, y, z\}$. Often it is convenient to consider the vortex distributions in terms of the sum and difference of the positive and negative vortex densities. The (total) *vortex density* is the sum of the positive and negative vortex densities

$$V(x, y, z) = n_{\mathrm{p}}(x, y, z) + n_{\mathrm{n}}(x, y, z) \tag{1}$$

and the *topological charge density* is the positive vortex density minus the negative vortex density

$$T(x, y, z) = n_{\mathrm{p}}(x, y, z) - n_{\mathrm{n}}(x, y, z). \tag{2}$$

These are the main vortex distributions. There are also other related quantities, some of which are discuss below.

2. TOOLS OF STOCHASTIC SINGULAR OPTICS

The properties of vortex distributions in stochastic optical fields are studied with the aid of numerical simulation and statistical optics calculations. Experimental work in this field is challenging because of the difficulty to identify and track numerous vortices in a stochastic optical field. Numerical simulations provide a powerful means to investigate the evolution of stochastic optical fields, because it can be applied for all types of stochastic optical fields. Statistical optics calculations are mainly used in the case of normally distributed optical fields.

2.1 Numerical simulations

The general procedure for numerical simulations is to start with an input field, which consists of a two-dimensional array of complex sampled values that represent the optical field on the transverse input plane. This field would be a (quasi) random function that may contain some non-trivial correlations. The input field is then propagated over a varying propagation distance, using a standard beam propagation algorithm. One such algorithm consists of the following three steps: 1) compute the Fourier transform of the input field (using an FFT routine), 2) multiply this spectrum with the appropriate propagation phase function, 3) compute the inverse Fourier transform of this result. The resulting two-dimensional array represents the optical field at some given propagation distance. One can now extract the locations of all the vortices in this optical field and use this to compute the various vortex distributions or other related quantities. This is done repeatedly for different input fields to compute an ensemble average of the vortex distributions. It is also done for different propagation distances to determine the evolution of the quantities under consideration as a function of propagation distance.

By using such numerical simulations, some unusual behaviour have been observed that cannot be explained by any current understanding of optical fields. For instance, in the case where the continuous phase of a speckle field is removed with an adaptive optical system[22–24] one observes the evolution of the vortex density as shown in Fig. 1. The fact that the scintillation index in Fig. 1(a) does not remain equal to 1 for all propagation distances indicates that the optical field is not in general normally distributed. One can understand the lower vortex density at large propagation distances compared to the initial vortex density in Fig. 1(b) as the result of a deconvolution process, but the reason for the dip in the vortex density at intermediate propagation distances is an unexplained phenomenon, which indicates that a new physical theory is required to explain the evolution of these stochastic optical fields.

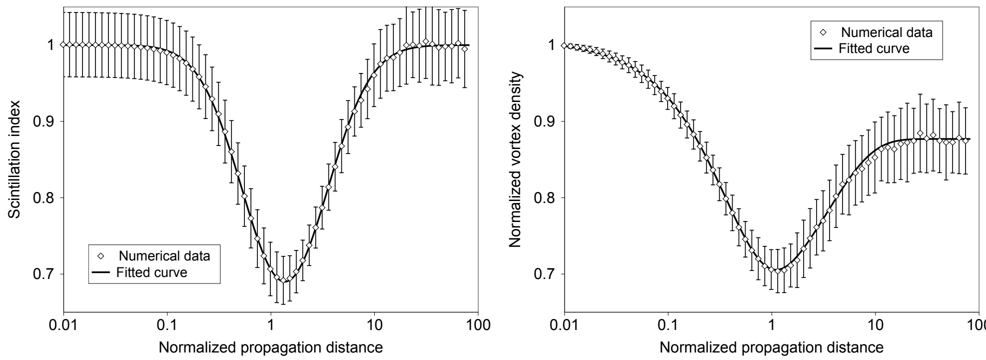

Figure 1. (a) Scintillation index and (b) normalized optical vortex density for a phase corrected speckle field, shown as a function of logarithmic normalized propagation distance. The diamonds represent numerical data, averaged over more than a hundred different simulations. The error bars represent standard deviations. The solid curve is a fitted curve through these data points.

In the case where the stochastic optical field is inhomogeneous, yet normally distributed, one can observe non-trivial variations in the spatial Fourier components of the vortex density and the topological charge density.[18, 19] Examples of such behaviour are shown in Fig. 2(a) and (b) for the vortex density and the topological charge density, respectively. These observation also represent cases that cannot be explained with existing theories.

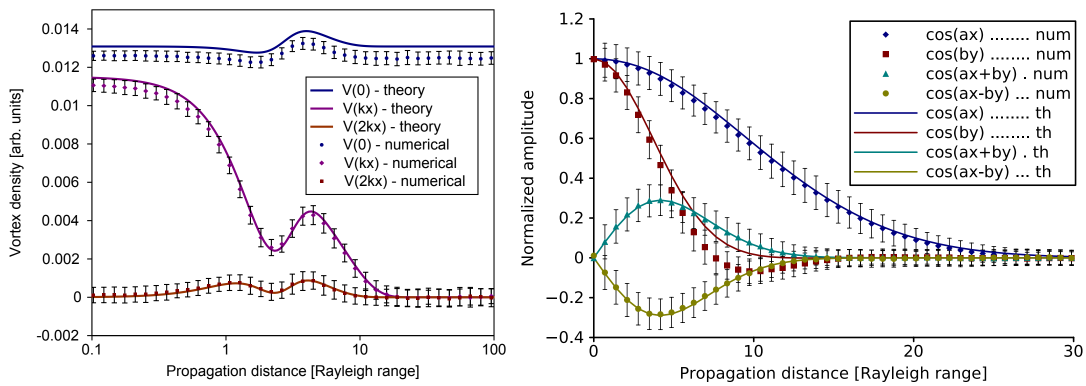

Figure 2. Variations in the spatial Fourier components of (a) the vortex density and (b) the topological charge density as a function of the normalized propagation distance for different initial stochastic optical field. The markers represent numerical data, averaged over more than a several different simulations. The error bars represent standard deviations. The solid curve represents statistical optical predictions in (a) and a numerical solution of a differential equation in (b).

2.2 Statistical optics calculations

Another tool that is useful for the analysis of stochastic optical fields that are normally distributed is statistical optical calculations. Although one can in principle also perform such calculations for stochastic optical fields that are not normally distributed, such calculations are in general not tractable.

The calculation of the expectation values of local quantities can be separated into two parts. One part is to obtain an expression for the local quantity in terms of local field correlation functions. This part can be done once and would be valid for all stochastic optical fields. The other part is to compute the local field correlation functions for a particular stochastic optical field, which are then substituted into the general expression for the local quantity to obtain its expression for a particular stochastic optical field.

2.3 General local two-point correlation functions

It is convenient to use the non-local two-point correlation function, also called the *mutual coherence function*,[25, 26]

$$\Gamma_{\rm full}(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2) = \langle g(\mathbf{x}_1, t_1)\overline{g}(\mathbf{x}_2, t_2)\rangle \tag{3}$$

to compute local two-point correlation functions. In Eq. (3), $\mathbf{x} = \{x, y, z\}$, $g(\cdot)$ is the complex scalar optical field, $\overline{g}(\cdot)$ is its complex conjugate and $\langle\cdot\rangle$ denotes the expectation value of a quantity. We assume that the optical field is monochromatic. Hence, the time dependence is unimportant and can be dropped. We also consider the correlation functions on a plane with the same z-coordinate for both points. hence,

$$\Gamma(x_1, y_1, x_2, y_2, z) = \langle g(x_1, y_1, z)\overline{g}(x_2, y_2, z)\rangle. \tag{4}$$

As an example, the intensity is obtained by setting $x_2 = x_1 = x$ and $y_2 = y_1 = y$, giving

$$I(x, y, z) = \langle g\,\overline{g}\rangle = \Gamma(x, y, x, y, z). \tag{5}$$

In the case of local correlation functions between the optical field and derivatives of the optical field, such as

$$\langle g\,\overline{g}_x\rangle = \langle g(x, y, z)\partial_x\overline{g}(x, y, z)\rangle, \tag{6}$$

the mutual coherence function serves as a generating function. Since the derivative and the expectation value both represent linear operations, they can be interchanged, provided that the correlation is done non-locally prior to the differential operation and only made local afterward. For example,

$$\langle g\,\overline{g}_x\rangle = \left[\partial_x\langle g(u, v, z)\overline{g}(x, y, z)\rangle\right]_{u=x,v=y} = \left[\partial_x\Gamma(u, v, x, y, z)\right]_{u=x,v=y}. \tag{7}$$

2.4 Expectation values of local quantities

The expectation value of a quantity $W(\mathbf{q})$, which is a function of N random variables $\mathbf{q} = \{q_n\}$, is

$$\langle W\rangle = \int W(\mathbf{q})P_{\mathbf{q}}(\mathbf{q})\ \mathrm{d}^N q, \tag{8}$$

where $P_{\mathbf{q}}(\mathbf{q})$ represents the joint probability density function for all the q_n's. Here the q_n's replace the optical field and its derivatives. If we restrict ourselves to correlation functions involving the optical field and its first derivatives, which we combine into a vector

$$\mathbf{G} = \left[g(\mathbf{x}), \partial_x g(\mathbf{x}), \partial_y g(\mathbf{x})\right]^T, \tag{9}$$

then $N = 6$ in Eq. (8) and there are 9 correlation functions, which are combined into a covariance matrix,

$$\langle \mathbf{G}\mathbf{G}^\dagger\rangle = M_1 = \begin{bmatrix} \langle g\overline{g}\rangle & \langle g_x\overline{g}\rangle & \langle g_y\overline{g}\rangle \\ \langle g\overline{g}_x\rangle & \langle g_x\overline{g}_x\rangle & \langle g_y\overline{g}_x\rangle \\ \langle g\overline{g}_y\rangle & \langle g_x\overline{g}_y\rangle & \langle g_y\overline{g}_y\rangle \end{bmatrix}. \tag{10}$$

The joint probability density function, in this case, is given by[3, 15, 18]

$$P_{\mathbf{q}}(\mathbf{q}) = \frac{\exp\left(-\mathbf{Q}^\dagger M_1^{-1}\mathbf{Q}\right)}{\pi^3\det(M_1)}, \tag{11}$$

where the q_n's are combined into a complex vector $\mathbf{Q} = \left[q_1 + \mathrm{i}q_2, q_3 + \mathrm{i}q_4, q_5 + \mathrm{i}q_6\right]^T$. The details of the function $W(\mathbf{q})$ depends on the specific quantity under investigation. Examples are discussed below.

2.5 Optical vortex density

The expectation value of the vortex density is given by the local average of the number of first order zero's in the optical field. This is given by[3, 15]

$$V(\mathbf{x}) = \int \left[P_{\mathbf{q}}(\mathbf{q})\right]_{q_1=q_2=0} |q_3 q_6 - q_4 q_5|\ \mathrm{d}^4 q. \tag{12}$$

The four integrations over q_3, q_4, q_5 and q_6, gives an expression for the vortex density

$$V(\mathbf{x}) = \frac{2\langle g\overline{g}\rangle\det(M_1) - B^2}{2\pi\langle g\overline{g}\rangle^2\sqrt{4\langle g\overline{g}\rangle\det(M_1) - B^2}}, \tag{13}$$

where

$$\begin{aligned}\det(M_1) &= \langle g\overline{g}\rangle(\langle g_x\overline{g}_x\rangle\langle g_y\overline{g}_y\rangle - \langle g_y\overline{g}_x\rangle\langle g_x\overline{g}_y\rangle) - \langle g_x\overline{g}_x\rangle\langle g_y\overline{g}\rangle\langle g\overline{g}_y\rangle \\ &\quad -\langle g_y\overline{g}_y\rangle\langle g_x\overline{g}\rangle\langle g\overline{g}_x\rangle + \langle g_x\overline{g}_y\rangle\langle g_y\overline{g}\rangle\langle g\overline{g}_x\rangle + \langle g_y\overline{g}_x\rangle\langle g_x\overline{g}\rangle\langle g\overline{g}_y\rangle) \qquad (14) \\ B &= \langle g\overline{g}\rangle(\langle g_x\overline{g}_y\rangle - \langle g_y\overline{g}_x\rangle) + \langle g\overline{g}_x\rangle\langle g_y\overline{g}\rangle - \langle g_x\overline{g}\rangle\langle g\overline{g}_y\rangle. \qquad (15)\end{aligned}$$

It is interesting to note that the vortex density can be expressed in terms of three quantities: the average intensity $\langle g\overline{g}\rangle$, the determinant of the covariance matrix $\det(M_1)$ given in Eq. (10), and another quantity B. The expression for the vortex density in Eq. (13) is valid for all stochastic optical fields. After calculating the field correlation functions for a particular stochastic optical field from the mutual coherence function, one can substitute them into Eq. (13) to obtain the vortex density for that particular case.

2.6 Topological charge density

The integral for the expectation value of the topological charge density is very similar to that of the vortex density in Eq. (12). It only differs in that the sign of the Jacobian is retained,

$$T(\mathbf{x}) = \int P_{\mathbf{q}}(\mathbf{q})|_{q_1=q_2=0} (q_3 q_6 - q_4 q_5)\ \mathrm{d}^4 q. \qquad (16)$$

After evaluating the four integrations over q_3, q_4, q_5 and q_6, one obtains an expression for the topological charge density in terms of the field correlation functions, given by

$$T(\mathbf{x}) = \frac{\mathrm{i}B}{2\pi\langle g\overline{g}\rangle^2}, \qquad (17)$$

where B is given in Eq. (15).

2.7 Local phase gradient

The expectation value of the local phase gradient gives an indication of the local spatial frequency of the field.[19] For an optical field $g(\mathbf{x}) = A(\mathbf{x})\exp[\mathrm{i}\theta(\mathbf{x})]$, the phase is given by

$$\theta(\mathbf{x}) = \frac{-\mathrm{i}}{2}\ln\left[\frac{g(\mathbf{x})}{g^*(\mathbf{x})}\right]. \qquad (18)$$

The gradient of the phase then becomes

$$\nabla\theta(\mathbf{x}) = -\mathrm{i}\frac{\overline{g}(\mathbf{x})\nabla g(\mathbf{x}) - g(\mathbf{x})\nabla\overline{g}(\mathbf{x})}{2|g(\mathbf{x})|^2}. \qquad (19)$$

The associated W-function that represents the local phase gradient is given by

$$W(\mathbf{q}) = \frac{(q_3 q_2 - q_4 q_1)\hat{x} + (q_5 q_2 - q_6 q_1)\hat{y}}{q_1^2 + q_2^2}. \qquad (20)$$

In this case one does not set q_1 and q_2 to zero, but instead the integration is done over all 6 q's. The resulting expression for the expectation value of the local phase gradient is given by

$$\mathbf{F}(\mathbf{x}) = \frac{\mathrm{i}[(\langle g\overline{g}_x\rangle - \langle g_x\overline{g}\rangle)\hat{x} + (\langle g\overline{g}_y\rangle - \langle g_y\overline{g}\rangle)\hat{y}]}{2\langle g\overline{g}\rangle} = \frac{\mathbf{v}_2}{2\langle g\overline{g}\rangle}. \qquad (21)$$

2.8 Magnitude of the local phase gradient

The expectation value of the magnitude of the local phase gradient can be computed with a W-function that represents the magnitude of the W-function for the local phase gradient given in Eq. (20). The W-function for the magnitude of the local phase gradient is given by

$$W(\mathbf{q}) = \frac{\sqrt{(q_3 q_2 - q_4 q_1)^2 + (q_5 q_2 - q_6 q_1)^2}}{q_1^2 + q_2^2}. \tag{22}$$

Although it is conceptually very similar to the local phase gradient, the resulting expression is quite different. After substituting Eq. (22) into Eq. (8) and evaluating the 6 q-integrals, one obtains

$$F(\mathbf{x}) = \frac{\sqrt{G+H}}{8\pi\sqrt{2}\langle g\overline{g}\rangle} \mathrm{E}\left(\sqrt{\frac{2H}{G+H}}\right), \tag{23}$$

where $\mathrm{E}(\cdot)$ is the complete elliptic integral of the second kind[27] and

$$G = 4\langle g\overline{g}\rangle(\langle g_x\overline{g}_x\rangle + \langle g_y\overline{g}_y\rangle) - (\langle g_x\overline{g}\rangle + \langle g\overline{g}_x\rangle)^2 - (\langle g_y\overline{g}\rangle + \langle g\overline{g}_y\rangle)^2 \tag{24}$$

$$\begin{aligned} H = \Big(& 16\langle g\overline{g}\rangle^2 \left[(\langle g_x\overline{g}_x\rangle - \langle g_y\overline{g}_y\rangle)^2 + (\langle g_x\overline{g}_y\rangle + \langle g_y\overline{g}_x\rangle)^2\right] \\ & -8\langle g\overline{g}\rangle \left\{\left[(\langle g_x\overline{g}\rangle + \langle g\overline{g}_x\rangle)^2 - (\langle g_y\overline{g}\rangle + \langle g\overline{g}_y\rangle)^2\right](\langle g_x\overline{g}_x\rangle - \langle g_y\overline{g}_y\rangle) \right. \\ & \left. +2(\langle g_x\overline{g}_y\rangle + \langle g_y\overline{g}_x\rangle)(\langle g\overline{g}_y\rangle + \langle g_y\overline{g}\rangle)(\langle g_x\overline{g}\rangle + \langle g\overline{g}_x\rangle)\right\} \\ & + \left[(\langle g_x\overline{g}\rangle + \langle g\overline{g}_x\rangle)^2 + (\langle g_y\overline{g}\rangle + \langle g\overline{g}_y\rangle)^2\right]^2\Big)^{1/2}. \end{aligned} \tag{25}$$

It is clear that $F(\mathbf{x}) \neq |\mathbf{F}(\mathbf{x})|$. The reason is that, in general, $\langle |g|\rangle \neq |\langle g\rangle|$.

2.9 Null crossing line density

Optical vortices appear where the null crossings from the real and imaginary parts of the optical field cross each other. Hence, the presence and characteristics of optical vortices are closely related to the presence of the null crossings in the real and imaginary parts of the optical field.

One needs to keep in mind that there is no unique way to separate the optical field into real and imaginary parts, since one can always multiply the optical field with a global phase factor without changing it. The quantities related to the null crossings in the real and imaginary parts of the optical field must be invariant to such a global phase factor. The line density of null crossings, which is defined by

$$N(\mathbf{x}) = \frac{1}{A}\int_A \delta(G_q)\sqrt{(\partial_x G_q)^2 + (\partial_y G_q)^2}\, \mathrm{d}x\mathrm{d}y, \tag{26}$$

where G_q represents either G_r or G_i, is invariant with respect to a global phase factor. Following a similar procedure as was done for the vortex density, one can write this integral in the form given in Eq. (8), but the integration is restricted to either $\{q_1, q_3, q_5\}$ or $\{q_2, q_4, q_6\}$ for the real or imaginary part of the optical field, respectively, with the remaining q's excluded from the expression. After evaluating the three q-integrals, one obtains

$$N(\mathbf{x}) = \frac{\sqrt{G+H}}{2\pi\sqrt{2}\langle g\overline{g}\rangle} \mathrm{E}\left(\sqrt{\frac{2H}{G+H}}\right), \tag{27}$$

for both the real and imaginary parts of the optical field. The result is 4 times the expectation value of the magnitude of the local phase gradient given in Eq. (23). The density of null crossings is the inverse of the average distance between null crossings. On the other hand, the magnitude of the local phase gradient is the inverse of the average distance of a phase cycle. Every phase cycle produces four null crossings. Therefore the average distance of a phase cycle is four times the average distance between null crossings.

2.10 Quantities with higher derivatives

Some local quantities are defined in terms of field correlation functions of optical fields with up to two derivatives. For these cases the covariance matrix expands to one with rank 6,

$$M_2 = \begin{bmatrix} \langle g\overline{g}\rangle & \langle g_x\overline{g}\rangle & \langle g_y\overline{g}\rangle & \langle g_{xx}\overline{g}\rangle & \langle g_{xy}\overline{g}\rangle & \langle g_{yy}\overline{g}\rangle \\ \langle g\overline{g}_x\rangle & \langle g_x\overline{g}_x\rangle & \langle g_y\overline{g}_x\rangle & \langle g_{xx}\overline{g}_x\rangle & \langle g_{xy}\overline{g}_x\rangle & \langle g_{yy}\overline{g}_x\rangle \\ \langle g\overline{g}_y\rangle & \langle g_x\overline{g}_y\rangle & \langle g_y\overline{g}_y\rangle & \langle g_{xx}\overline{g}_y\rangle & \langle g_{xy}\overline{g}_y\rangle & \langle g_{yy}\overline{g}_y\rangle \\ \langle g\overline{g}_{xx}\rangle & \langle g_x\overline{g}_{xx}\rangle & \langle g_y\overline{g}_{xx}\rangle & \langle g_{xx}\overline{g}_{xx}\rangle & \langle g_{xy}\overline{g}_{xx}\rangle & \langle g_{yy}\overline{g}_{xx}\rangle \\ \langle g\overline{g}_{xy}\rangle & \langle g_x\overline{g}_{xy}\rangle & \langle g_y\overline{g}_{xy}\rangle & \langle g_{xx}\overline{g}_{xy}\rangle & \langle g_{xy}\overline{g}_{xy}\rangle & \langle g_{yy}\overline{g}_{xy}\rangle \\ \langle g\overline{g}_{yy}\rangle & \langle g_x\overline{g}_{yy}\rangle & \langle g_y\overline{g}_{yy}\rangle & \langle g_{xx}\overline{g}_{yy}\rangle & \langle g_{xy}\overline{g}_{yy}\rangle & \langle g_{yy}\overline{g}_{yy}\rangle \end{bmatrix} . \tag{28}$$

The calculations of these quantities are in general extremely complex. An indication of this complexity is the fact that, while the determinant of the covariance matrix in Eq. (10) has 6 terms, the determinant of the covariance matrix in Eq. (28) has 720 terms.

These quantities are also calculated with an expression similar to the one in Eq. (8), but with $N = 12$. Hence,

$$\mathbf{Q} = [q_1 + \mathrm{i}q_2, q_3 + \mathrm{i}q_4, q_5 + \mathrm{i}q_6, q_7 + \mathrm{i}q_8, q_9 + \mathrm{i}q_{10}, q_{11} + \mathrm{i}q_{12}]^T . \tag{29}$$

Examples of the expectation values of local quantities that require such higher derivatives are the distributions of the Poincarè-Hopf indices.[28] Field correlation functions consisting of optical fields with more than one derivative also appears when derivatives of local quantities are computed. This is the case when these local quantities appear in dynamical equations. These higher derivative quantities are not discussed here.

3. ADDRESSING THE COMPLEXITY IN STOCHASTIC SINGULAR OPTICS

One of the main challenges in statistical optics calculations in stochastic singular optics is the severe complexity that one often encounters. The expressions for the expectation values of local quantities, such as the topological charge density and the vortex density, contain multivariate polynomials in terms of the field correlation functions. These expressions can be rather complicated and when differential operations are performed on them the results become even more complicated. With the aid of coordinate invariance one can simplify the expressions and identify the expectation value of local quantities.

3.1 Coordinate rotation

The choice of coordinate system is arbitrary and if one rotates the x- and y-axes of the coordinate system by an arbitrary angle, the expression of the local expectation value of a quantity should not change. This implies that the polynomials in the expressions of the local expectation values of quantities must be invariant (or covariant) with respect to such coordinate transformations. Rotations in the (x, y)-plane can be represented by the elements of the SO(2) Lie group.* The invariant polynomials are singlets of the SO(2) Lie group.

3.2 Transformations of field correlation functions

Under the SO(2) Lie group of rotations in the (x, y)-plane the covariant matrix in Eq. (10) transforms as

$$M_1 \to OM_1O^{-1}, \tag{30}$$

where

$$O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} , \tag{31}$$

is a reducible element of the SO(2) Lie group, with α being the rotation angle, and $O^{-1} = O^T$.

*Since the SO(2) Lie group is isomorphic to the U(1) Lie group, one can also use the latter. However, for our purposes here it is more convenient to consider the transformation in terms of two-dimensional real-valued matrices instead of one-dimensional complex-valued factors.

Since Eq. (31) is reducible, the transformations that it imposes on the individual field correlation functions can be divided into different groups that transform differently. One can extract the transformation rules for the individual field correlation functions in these different groups from Eqs. (10), (30) and (31). The first group has just one element, the intensity, which does not transform at all, $\langle g\overline{g}\rangle \rightarrow \langle g\overline{g}\rangle$. The next group transforms as a doublet under SO(2)

$$\begin{aligned} \langle g\overline{g}_x\rangle &\rightarrow \langle g\overline{g}_x\rangle\cos(\alpha) - \langle g\overline{g}_y\rangle\sin(\alpha) \\ \langle g\overline{g}_y\rangle &\rightarrow \langle g\overline{g}_x\rangle\sin(\alpha) + \langle g\overline{g}_y\rangle\cos(\alpha). \end{aligned} \tag{32}$$

To be invariant under the coordinate rotation [singlets of the SO(2) Lie group] the multivariate polynomials must consist of combinations of field correlation functions, such that their transformations cancel off any additional terms that are formed. We denote all these singlets as τ_n's. For example $\tau_0 = \langle g\overline{g}\rangle$. The topological charge density can now be expressed in terms of these singlets, giving a much simpler expression

$$T = \frac{\tau_0\tau_2 - \tau_6}{2\pi\tau_0^2}. \tag{33}$$

4. RELATIONSHIPS AMONG LOCAL QUANTITIES

To demonstrate the benefit of using orthogonal singlets, we discuss here a number of relationships that exist among the local quantities.

4.1 Vortex density and related quantities

The expressions of the vortex density in terms of SO(2) singlets is given by

$$V(\mathbf{x}) = \frac{2\tau_0(\tau_0\tau_7 - \epsilon_1) + (\tau_0\tau_2 - \tau_6)^2}{2\pi\tau_0^2\sqrt{4\tau_0(\tau_0\tau_7 - \epsilon_1) + (\tau_0\tau_2 - \tau_6)^2}} \tag{34}$$

where

$$\epsilon_1 = \langle g_x\overline{g}_x\rangle\langle g_y\overline{g}\rangle\langle g\overline{g}_y\rangle + \langle g_y\overline{g}_y\rangle\langle g_x\overline{g}\rangle\langle g\overline{g}_x\rangle - \langle g_x\overline{g}_y\rangle\langle g_y\overline{g}\rangle\langle g\overline{g}_x\rangle - \langle g_y\overline{g}_x\rangle\langle g_x\overline{g}\rangle\langle g\overline{g}_y\rangle. \tag{35}$$

We notice that certain combinations that appear in Eq. (34) also appear in other local quantities. For instance, comparing Eqs. (33) and (34), one notices that the same combination $(\tau_0\tau_2 - \tau_6)$ appears in both. The determinant of the covariance matrix, which appears in the expression for the vortex density in Eq. (13), can be expressed as

$$\det(M_1) = \tau_0\tau_7 - \epsilon_1. \tag{36}$$

The vortex density can therefore be simplified to

$$V = \frac{Q + 2T^2}{2\sqrt{Q + T^2}} = \frac{1}{2}\sqrt{Q + T^2} + \frac{T^2}{2\sqrt{Q + T^2}}, \tag{37}$$

where T is the topological charge density, given in Eq. (33), and for convenience, we combined the determinant of the covariance matrix and the intensity τ_0 into a quantity Q, given by

$$Q = \frac{\det(M_1)}{\pi^2\tau_0^3}. \tag{38}$$

4.2 Curl of the local phase gradient

There exists a relationship between the local phase gradient and the topological charge density

$$\nabla \times \mathbf{F}(\mathbf{x}) = 2\pi\ T(\mathbf{x})\hat{z}. \tag{39}$$

One can verify Eq. (39) by using the following identities among singlets

$$\begin{aligned} \nabla\times\mathbf{v}_2 &= 2\tau_2\hat{z} = \mathrm{i}2(\langle g_x\overline{g}_y\rangle - \langle g_y\overline{g}_x\rangle)\hat{z} &(40)\\ \nabla\tau_0 &\equiv \mathbf{v}_1 = (\langle g_x\overline{g}\rangle + \langle g\overline{g}_x\rangle)\hat{x} + (\langle g_y\overline{g}\rangle + \langle g\overline{g}_y\rangle)\hat{y} &(41)\\ \mathbf{v}_1\times\mathbf{v}_2 &= 2\tau_6\hat{z} = \mathrm{i}2(\langle g_x\overline{g}\rangle\langle g\overline{g}_y\rangle - \langle g_y\overline{g}\rangle\langle g\overline{g}_x\rangle)\hat{z}. &(42)\end{aligned}$$

Applying the curl to $\mathbf{F}(\mathbf{x})$, we obtain

$$\begin{aligned}\nabla\times\mathbf{F}(\mathbf{x}) &= \nabla\times\left(\frac{\mathbf{v}_2}{2\tau_0}\right) = \frac{\nabla\times\mathbf{v}_2}{2\tau_0} - \frac{\nabla\tau_0\times\mathbf{v}_2}{2\tau_0^2} = \frac{\tau_2\hat{z}}{\tau_0} - \frac{\mathbf{v}_1\times\mathbf{v}_2}{2\tau_0^2}\\ &= \frac{(\tau_0\tau_2-\tau_6)\hat{z}}{\tau_0^2} = 2\pi\ T(\mathbf{x})\hat{z}. \qquad (43)\end{aligned}$$

Hence, we confirmed that the relationship between the local phase gradient and the topological charge density, given in Eq. (39), is valid for stochastic optical fields.

4.3 Intensity transport

Applying the longitudinal derivative to the intensity $\langle g\overline{g}\rangle = \tau_0$, one obtains

$$\partial_z\tau_0 = \frac{-\mathrm{i}}{2k}\left(\langle g_{xx}\overline{g}\rangle - \langle g\overline{g}_{xx}\rangle + \langle g_{yy}\overline{g}\rangle - \langle g\overline{g}_{yy}\rangle\right) = \frac{\nabla\cdot\mathbf{v}_2}{2k}. \qquad (44)$$

Since $\mathbf{v}_2$ appears in the local phase gradient, we consider the divergence of the local phase gradient,

$$\nabla\cdot\mathbf{F} = \nabla\cdot\left(\frac{\mathbf{v}_2}{2\tau_0}\right) = \frac{\nabla\cdot\mathbf{v}_2}{2\tau_0} - \frac{\nabla\tau_0\cdot\mathbf{v}_2}{2\tau_0^2} = \frac{k\partial_z\tau_0}{\tau_0} - \frac{\nabla\tau_0\cdot\mathbf{F}}{\tau_0}, \qquad (45)$$

where we substituted Eq. (44). Hence,

$$k\partial_z\tau_0 = \tau_0\nabla\cdot\mathbf{F} + (\nabla\tau_0)\cdot\mathbf{F} = \nabla\cdot(\tau_0\mathbf{F}). \qquad (46)$$

The expression in Eq. (46) is a dynamical equation that relates the intensity and the local phase gradient. It is a known result[29] for deterministic optical fields, often referred to as the *intensity transport equation*. Here we see that the equation also applies for stochastic optical fields.

5. CONCLUSIONS

We reviewed some of the recent progress in stochastic singular optics and provided some results from numerical simulations and statistical optics calculations. These results include cases that reveal phenomena that are yet to be understood. Using statistical optics calculations, we obtained general expressions in terms of the field correlation functions for the vortex density, the topological charge density, the phase gradient, the magnitude of the phase gradient and the null-crossing line density. These expressions are often rather complex. One can mitigate this complexity by exploiting the invariance of these quantities with respect to coordinate rotations, which form an SO(2) group. All the quantities of interest can be expressed as singlets of the SO(2) group. In terms of these SO(2) singlets the expressions for the above-mentioned quantities become significantly simpler. In some cases the simpler expressions reveal algebraic relationships among these quantities. Using identities that relate the different singlets, we also obtained differential relationships among some of the quantities.

REFERENCES

[1] Dainty, J. C., "The statistics of speckle patterns," *Prog. Opt.* **14**, 1–46 (1977).

[2] Goodman, J. W., "Statistical properties of laser speckle patterns," in [*Laser speckle and related phenomena*], 9–75, Springer (1975).

[3] Berry, M. V., "Disruption of wavefronts: statistics of dislocations in incoherent gaussian random waves," *J. Phys. A: Math. Gen.* **11**, 27–37 (1978).

[4] Cullet, P., Gil, L., and Rocca, F., "Optical vortices," *Opt. Commun.* **73**, 403–408 (1989).

[5] Allen, L., Padgett, M. J., and Babiker, M., "The orbital angular momentum of light," in [*Prog. Opt.*], **39**, 291–372, Elsevier (1999).

[6] Soskin, M. S. and Vasnetsov, M. V., "Singular optics," *Prog. Opt.* **42**, 219–276 (2001).

[7] Dennis, M. R., O'Holleran, K., and Padgett, M. J., "Singular optics: optical vortices and polarization singularities," *Prog. Opt.* **53**, 293–363 (2009).

[8] Baranova, N. B., Zel'dovich, B. Y., Mamaev, A. V., Pilipetskiĭ, N., and Shkunov, V. V., "Dislocations of the wavefront of a speckle-inhomogeneous field (theory and experiment)," *Soviet Journal of Experimental and Theoretical Physics Letters* **33**, 195–199 (1981).

[9] Freund, I., Shvartsman, N., and Freilikher, V., "Optical dislocation networks in highly random media," *Opt. Commun.* **101**, 247–264 (1993).

[10] Freund, I., "Optical vortices in gaussian random wave fields: statistical probability densities," *J. Opt. Soc. Am. A* **11**, 1644–1652 (1994).

[11] Shvartsman, N. and Freund, I., "Vortices in random wave fields: nearest neighbor anticorrelations," *Phys. Rev. Lett.* **72**, 1008–1011 (1994).

[12] Freund, I. and Shvartsman, N., "Wave-field phase singularities: the sign principle," *Phys. Rev. A* **50**(6), 5164 (1994).

[13] Shvartsman, N. and Freund, I., "Speckle spots ride phase saddles sidesaddle," *Opt. Commun.* **117**(3), 228–234 (1995).

[14] Freund, I. and Wilkinson, M., "Critical-point screening in random wave fields," *J. Opt. Soc. Am. A* **15**(11), 2892–2902 (1998).

[15] Berry, M. V. and Dennis, M. R., "Phase singularities in isotropic random waves," *Proc. R. Soc. Lond. A* **456**, 2059–2079 (2000).

[16] Dennis, M., "Correlations and screening of topological charges in gaussian random fields," *J. Phys. A: Math. Gen.* **36**, 6611–6628 (2003).

[17] Roux, F. S., "Lateral diffusion of the topological charge density in stochastic optical fields," *Opt. Commun.* **283**, 4855–4858" (2010).

[18] Roux, F. S., "Anomalous transient behavior from an inhomogeneous initial optical vortex density," *J. Opt. Soc. Am. A* **28**, 621–626 (2011).

[19] Roux, F. S., "Lateral phase drift of the topological charge density in stochastic optical fields," *Optics Communications* **285**, 947 – 952 (2012).

[20] Mercier, R. P., "Diffraction by a screen causing large random phase fluctuations," in [*Proc. Cambridge Philos. Soc.*], **58**, 382–400 (1962).

[21] Roux, F. S., "Distribution of angular momentum and vortex morphology in optical beams," *Opt. Comm.* **242**, 45–55 (2004).

[22] Chen, M. and Roux, F. S., "Influence of the least-squares phase on optical vortices in strongly scintillated beams," *Phys. Rev. A* **80**, 013824 (2009).

[23] Chen, M. and Roux, F. S., "Evolution of the scintillation index and the optical vortex density in speckle fields after removal of the least-squares phase," *J. Opt. Soc. Am. A* **27**, 2138–2143 (2010).

[24] Chen, M., Dainty, C., and Roux, F. S., "Speckle evolution with multiple steps of least-squares phase removal," *Phys. Rev. A* **84**, 023846 (2011).

[25] Goodman, J. W., [*Statistical optics*], Wiley-Interscience, New York (1985).

[26] Mandel, L. and Wolf, E., [*Optical coherence and quantum optics*] (1995).

[27] Abramowitz, M. and Stegun, I. A., [*Handbook of Mathematical Functions*], Dover, Toronto (1972).

[28] Freund, I. and Kessler, D., "Critical point trajectory bundles in singular wave fields," *Opt. Commun.* **187**, 71–90 (2001).

[29] Teague, M. R., "Irradiance moments: their propagation and use for unique retrieval of phase," *J. Opt. Soc. Am.* **72**, 1199–1209 (1982).