CAVALIERI INTEGRATION

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Abstract. We use Cavalieri's principle to develop a novel integration technique which we call *Cavalieri integration*. Cavalieri integrals differ from Riemann integrals in that non-rectangular integration strips are used. In this way we can use single Cavalieri integrals to find the areas of some interesting regions for which it is difficult to construct single Riemann integrals.

We also present two methods of evaluating a Cavalieri integral by first transforming it to either an equivalent Riemann or Riemann-Stieltjes integral by using special transformation functions $h(x)$ and its inverse $g(x)$, respectively. Interestingly enough it is often very difficult to find the transformation function $h(x)$, whereas it is very simple to obtain its inverse $g(x)$.

1. Introduction

We will use Cavalieri's principle to develop a novel integration technique which can be used to almost effortlessly find the area of some interesting regions for which it is rather difficult to construct single Riemann integrals. We will call this type of integration *Cavalieri integration*. As the name suggests, Cavalieri integration is based on the well known Cavalieri principle, stated here without proof [3]:

Theorem 1.1 (Cavalieri's principle)**.** *Suppose two regions in a plane are included between two parallel lines in that plane. If every line parallel to these two lines intersects both regions in line segments of equal length, then the two regions have equal areas.*

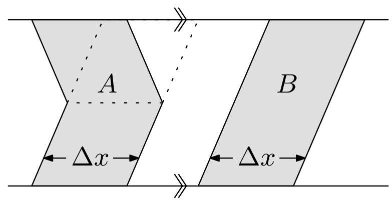

Figure 1. Simple illustration of Cavalieri's principle in $\mathbb{R}^2$, with area $A =$ area B.

Inspired by Cavalieri's principle, we pose the following question: *what happens when we replace the usual rectangular integration strip of the Riemann sum with an integration strip that has a non-rectangular shape*? It turns out that such a formulation leads to a consistent scheme of integration with a few surprising results.

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By considering non-rectangular integration strips we form a Cavalieri sum which can either be transformed to a normal Riemann sum (of an equivalent region) by using a transformation function $h(x)$, or to a Riemann-Stieltjes sum by using the inverse transformation function $g(x)$.

The main result of Cavalieri integration can be demonstrated by using a simple example. Consider the region bounded by the x-axis and the lines $f(x) = x$, $a(y) = 1 - y$ and $b(y) = 4 - y$, shown in Figure 2.A. Notice that we cannot express the area of this region as a single Riemann integral. We can however calculate the area of this region by using a *single* Cavalieri integral:

$$\text{Area} = \int_{a(y)}^{b(y)} f(x)\, dx,$$

which is related to a Riemann integral and a Riemann-Stieltjes integral as follows:

$$\int_{a(y)}^{b(y)} f(x)\, dx = \int_a^b f \circ h(x)\, dx = \int_{a'}^{b'} f(x)\, dg(x).$$

For the present example we have the following result, since $h(x) = x/2$ and $g(x) = 2x$:

$$\int_{1-y}^{4-y} x\, dx = \int_1^4 \frac{x}{2}\, dx = \int_{0.5}^{2} x\, d2x = 3.75.$$

The transformed regions $f \circ h(x)$ (corresponding to the Riemann formulation) and $f(x) \cdot g'(x)$ (corresponding to the Riemann-Stieltjes formulation) are shown in Figure 2.B and Figure 2.C, respectively.

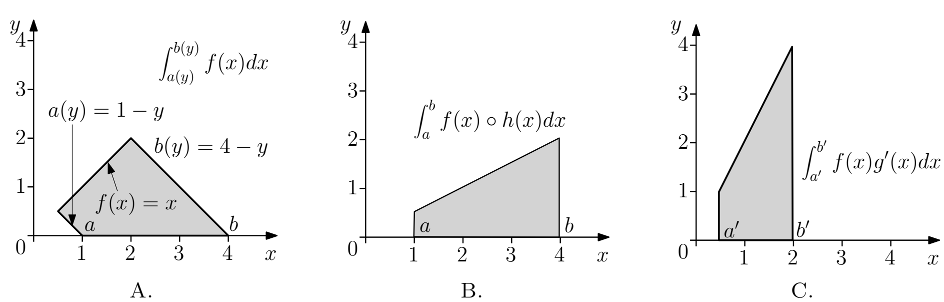


FIGURE 2. Illustration of Cavalieri integration by example.

In this paper we will show how to find the transformation function $h(x)$ and its inverse $g(x)$. We first give a brief overview of classical integration theory (Section 2), followed by the derivation of Cavalieri integration in Section 3. Finally we present a number of fully worked examples in Section 4, which clearly demonstrate how Cavalieri integration can be applied to a variety of regions.

2. Classical Integration Theory

One of the oldest techniques for finding the area of a region is the *method of exhaustion*, attributed to Antiphon [4]. The method of exhaustion finds the area of a region by inscribing inside it a sequence of polygons whose areas converge to the area of the region. Even though classical integration theory is a well established field there are still new results being added in modern times. For example, in the very interesting paper by Ruffa [5] the method of exhaustion was generalized, which lead to an integration formula that is valid for all Riemann integrable functions:

$$\int_a^b f(x)dx = (b-a)\sum_{n=1}^{\infty}\sum_{m=1}^{2^n-1}(-1)^{m+1}2^{-n}f\left(a+\frac{m(b-1)}{2^n}\right).$$

Classical integration theory is however very different from the method of exhaustion, and is mainly attributed to Newton, Leibniz and Riemann. Newton and Leibniz discovered the fundamental theorem of Calculus independently and developed the mathematical notation for classical integration theory. Riemann formalized classical integration by introducing the concept of limits to the foundations established by Newton and Leibniz. However, the true father of classical integration theory is probably Bonaventura Cavalieri (1598–1647).

Cavalieri devised methods for computing areas by means of 'indivisibles' [1]. In the method of indivisibles, a region is divided into infinitely many indivisibles, each considered to be both a one-dimensional line segment, and an infinitesimally thin two-dimensional rectangle. The area of a region is then found by summing together all of the indivisibles in the region. However, Cavalieri's method of indivisibles was heavily criticized due to the "indivisible paradox", described next [1].

2.1. **Indivisible paradox.** Consider a scalene triangle, ΔABC, shown in Figure 3.A. By dropping the altitude to the base of the triangle, ΔABC is partitioned into two triangles of unequal area. If both the left (ΔABD) and right (ΔBDC) triangles are divided into indivisibles then we can easily see that each indivisible (for example EF) in the left triangle corresponds to an equal indivisible (for example GH) in the right triangle. This would seem to imply that both triangles must have equal area!

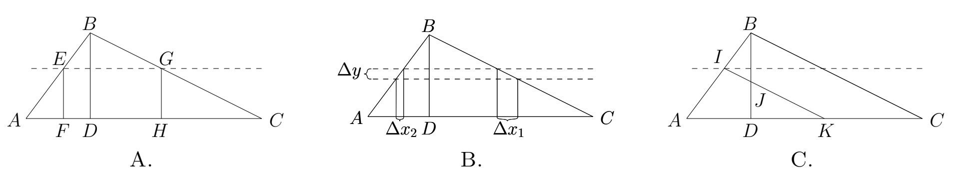


FIGURE 3. Cavalieri's indivisible paradox.

Of course this argument is clearly flawed. To see this, we can investigate it more closely from a measure-theoretic point of view, as shown in Figure 3.B. Drawing a strip of width Δy through the triangle and calculating the pre-image of this strip produces two intervals on the x-axis with unequal width. Letting $\Delta y \to 0$ produces the two indivisibles EF and GH. However, it does not matter how small you make Δy, the two interval lengths Δx_1 and Δx_2 will never be equal. In other words, the area that EF and GH contributes to the total area of the triangle must be different.

There is an even simpler way to renounce the above paradox: instead of using indivisibles parallel to the y-axis, we use indivisibles parallel to BC, as shown in Figure 3.C. Then each pair of corresponding indivisibles IJ in ΔABD and JK

in ΔBDC clearly has different lengths almost everywhere. Therefore the areas of ΔABD and ΔBDC need not be the same.

This trick of considering indivisibles (or infinitesimals) other than those parallel to the y-axis forms the basis of Cavalieri integration, in which non-rectangular integration strips will be used.

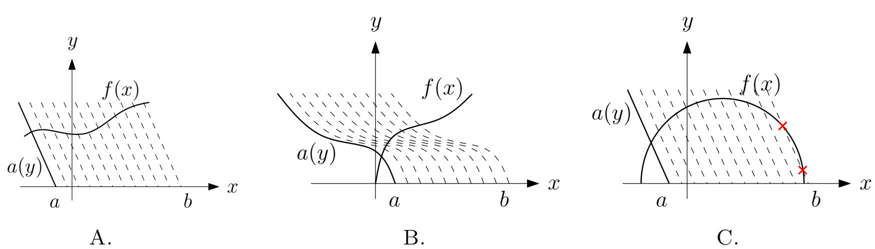
3. Cavalieri Integration

We present a method of integration which we will refer to as *Cavalieri integration*, in which the primary difference from ordinary Riemann integration is that more general integration strips can be used. In some sense the Cavalieri integral can also be seen as a generalization of the Riemann integral, in that the Cavalieri formulation reduces to the ordinary Riemann integral when the integration strips are rectangular. That is not to say that the Cavalieri integral extends the class of Riemann-integrable functions. In fact, the class of Cavalieri-integrable functions is exactly equivalent to the class of Riemann-integrable functions. However, the Cavalieri integral allows us to express the areas of some regions as single integrals for which we would have to write down multiple ordinary Riemann integrals.

3.1. **Preliminaries and Definitions.** In order to develop (and clearly present) the Cavalieri integration theory, a number of definitions must first be introduced. Also note that we will restrict our attention to integration in $\mathbb{R}^2$, with coordinate axes x and y.

Definition 3.1 (Translational function)**.** A continuous function $a(y)$ is called a translational function with respect to a continuous function $f(x)$ on the interval $[a, b]$ if $\{a \circ f(x) + z = x\}$ is singular, for every $z \in (b - a)$ and $a(0) = a$.

The above definition says that any continuous function $a(y)$ which intersects a continuous function $f(x)$ *exactly once* for an arbitrary translation on the x-axis within the interval $[a, b]$ is called a translational function. Two examples of translational functions are shown in Figure 4.A and Figure 4.B, and Figure 4.C presents an example of a linear function $a(y)$ which is not translational with respect to $f(x)$.



A. B. C.

Figure 4. Examples of translational, and non-translational functions.

Definition 3.2 (Cavalieri region R)**.** Let R be any region (in $\mathbb{R}^2$) bounded by a nonnegative function $f(x)$ (which is continuous on the interval $[a', b']$), the x-axis, and the boundary functions $a(y)$ and $b(y)$, where $a(y)$ is a translational function, $b(y) := a(y) + (b - a)$. Furthermore we have that a' and b' are the unique x-values for which $a(y)$ and $b(y)$ intersect $f(x)$, respectively; and $a = a(0)$ and $b = b(0)$. Then R is called the *Cavalieri region* bounded by $f(x)$,$a(y)$, $b(y)$ and the x-axis.

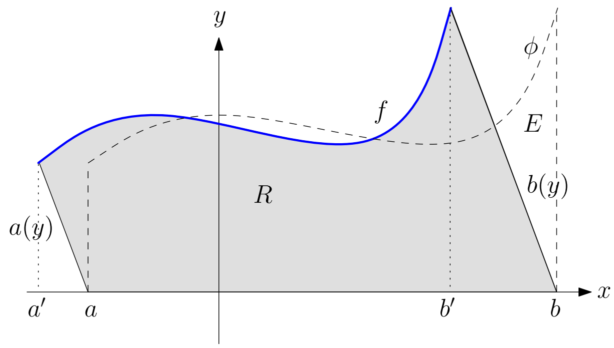


FIGURE 5. A Cavalieri region R with integration boundaries $a(y)$ and $b(y)$, and an equivalent region E with integration boundaries $x = a$ and $x = b$.

The Cavalieri integral (which we will formally define in Definition 3.20) can be related to an ordinary Riemann integral through a particular transformation h, which we will consider in some detail below. It may be useful to think of this transformation (at least intuitively) as transforming any Cavalieri region R into an equivalent region E with equal area (see Figure 5), but with integration boundaries $x = a$ and $x = b$. That is, the area of the equivalent region E can easily be expressed in terms of an ordinary definite integral $\int_a^b \phi(x)\, dx$.

Definition 3.3 (Transformation function h)**.** Let $a(y)$ be a translational function. The mapping $h : [a, b] \to [a', b']$, which maps $x_i^1 \in [a, b]$ to $x_i^2 \in [a', b']$, is defined as $h(x_i^1) := \{x_i^2 \in [a', b'] : a(f \circ x_i^2) + [x_i^1 - a] = x_i^2, a = a(0)\}$, which we will refer to as the transformation function (we will prove that it is indeed a function below).

Proposition 3.4. *The mapping $h : [a, b] \to [a', b']$ is a function.*

Proof. That h is a function follows directly from the definition of a translational function (Definition 3.1), since we know that $\{a \circ f(x_i^2) + [x_i^1 - a] = x_i^2\}$ must be singular for every $[x_i^1 - a] \in (b - a)$. That is, h maps every point $x_i^1 \in [a, b]$ to exactly one point $x_i^2 \in [a', b']$. □

Proposition 3.5. *The transformation function h is strictly monotone on $[a, b]$.*

Proof. Let R be a Cavalieri region bounded by $f(x), a(y), b(y)$ and the x-axis, as shown in Figure 6. Two possibilities may arise.

Case I: $\{a \circ f(x) = x\} = a' \Rightarrow h$ is strictly increasing:
Consider any translation of $a(y)$, $a(y) + \Delta c$, s.t. $a + \Delta c \in (a, b)$. Since the domain $\mathcal{D}(f) \geq a'$, and since $a(y)$ intersects $f(x)$ at a', the translation $a(y) + \Delta c$ cannot also intersect $f(x)$ at a'. Instead, we clearly have that $a(y) + \Delta c$ must intersect $f(x)$ at a point $c' > a'$ on $\mathcal{D}(f)$.

We now define A as the region bounded by the translational functions $a(y)$ and $a(y) + \Delta c$, and the lines $y = a'$ and $y = b'$ (see Figure 6). The continuous function $f(x)$ on the interval $[a', b']$ must lie within the region A, since any point of $f(x)$ outside of this region would imply that $a(y)$ cannot be a translational function. That is, if $f(x)$ has points outside of region A, then there exists a translation of $a(y)$ s.t. $a(y)$ intersects $f(x)$ at more than one point.

Now consider any translation of $a(y)$, $a(y)+\Delta d$, where $\Delta d > \Delta c$ and $\Delta d \in (a, b]$. Suppose that $a(y) + \Delta d$ induces a point d', with $a' < d' < c'$. That is, $a(y) + \Delta d$ intersects $f(x)$ at some point in region A.

The functions $a(y) + \Delta c$ and $a(y) + \Delta d$ are continuous on the interval $y \in [0, \gamma]$, where $\gamma := \{a \circ f(x) + \Delta d = x\}$ (in fact, any translational function must be continuous on $y \in \mathbb{R}$). Now let $\Psi := a(y) + \Delta c - \big(a(y) + \Delta d\big)$, which is again a continuous function on $[0, \gamma]$. Since $c < d$ and $c' > d'$ by assumption, it follows that $\Psi(0) < 0$ and $\Psi(\gamma) > 0$. From the intermediate value theorem it follows that there exists a point $\alpha \in [0, \gamma]$ s.t. $\Psi(\alpha) = 0$. That is, $a(\alpha) + \Delta c = a(\alpha) + \Delta d$. But this is impossible, since $a(y) + \Delta d$ is a translation of $a(y) + \Delta c$. Therefore $c' = h(c) < h(d) = d'$.

Since Δc is arbitrary and $d > c \Rightarrow h(d) > h(c)$, h is strictly increasing on $[a, b]$.

Case II: $\{a \circ f(x) = x\} = b' \Rightarrow h$ is strictly decreasing:
The second case can be proved in a similar manner as Case I above, in which case h is a strictly decreasing function with the order of the induced partition $\mathcal{P}_2$ reversed.

Since h is either strictly increasing (Case I) or strictly decreasing (Case II), it is strictly monotone on $[a, b]$. □

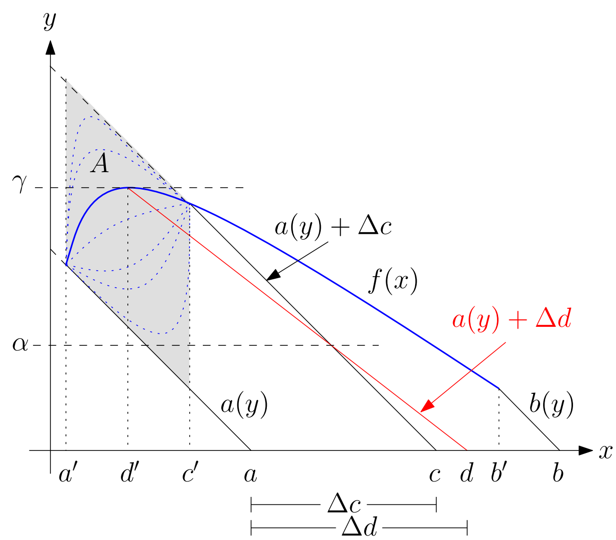


FIGURE 6. Sketch for the proof of Proposition 3.5.

Proposition 3.6. *The transformation function h is continuous on $[a, b]$.*

Proof. Choose an arbitrary value $x_*^1 \in [a, b]$ such that $x_*^1 = a+c$. We can now define a sequence (x_i^1) with $x_i^1 = x_*^1 + \frac{1}{i}, \forall\, i \in \mathbb{N}$. Now $(x_i^1) \to x_*^1$ as $i \to \infty$. The sequence of functions $\big(a(y) + [x_i^1 - x_*^1 + c]\big)$ has x-intercepts equal to (x_i^1). The mapping h now generates a new sequence (x_i^2) s.t. $\forall\, i$, $x_i^2 = \{x_i^2 : a \circ f(x_i^2) + [x_i^1 - x_*^1 + c] = x_i^2\}$.

Now taking the limit as $i \to \infty$

$$\begin{aligned}
\lim_{i\to\infty} x_i^2 &= \lim_{i\to\infty} [a \circ f(x_i^2) + (x_i^1 - x_*^1 + c) = x_i^2] \\
&= \lim_{i\to\infty} [a \circ f(x_i^2) + (\frac{1}{i} + c) = x_i^2] \\
&= [a \circ f(x_i^2) + (\lim_{i\to\infty} \frac{1}{i} + c) = x_i^2] \\
&= [a \circ f(x_i^2) + c) = f(x_i^2)] \\
&= [a(x_i^2) + [x_*^1 - a]) = f(x_i^2)] \\
&= x_*^2
\end{aligned}$$

This shows that $x_i^2 \to x_*^2$ as $x_i^1 \to x_*^1$ assuming $[a \circ f(x_i^2) + [x_*^1 - a]) = f(x_i^2)]$ has one unique solution, which must be the case since $a(y)$ is a translational function. The function h must be continuous at x_*^1 since $x_i^2 \to x_*^2$ as $x_i^1 \to x_*^1$. Since x_*^1 is arbitrary, h is a continuous function on $[a, b]$. □

Proposition 3.7. *The transformation function h is bijective on $[a, b]$.*

Proof. That h is injective on $[a, b]$ follows from the fact that h is strictly monotone on $[a, b]$ (by Proposition 3.5). Furthermore h is clearly surjective on $[a, b]$, since it is continuous on $[a, b]$ (by Proposition 3.6). Since h is both injective and surjective on $[a, b]$, h is also bijective on $[a, b]$. □

3.2. **Derivation of Cavalieri Integration.** Since we want to derive the Cavalieri integral – which uses more general integration strips than the rectangles of the Riemann integral – we first need to formally define valid integration strips.

Definition 3.8 (Integration strip)**.** An integration strip is an area bounded below by the x-axis, on the left by a translational function $a(y)$ w.r.t. $f(x)$ on $[a, b]$, from the right by $b(y) = a(y) + (b - a)$, and from above by the line $y = c$.

An example of three integration strips is given in Figure 7, where Figure 7.a corresponds to the usual Riemann integration strip.

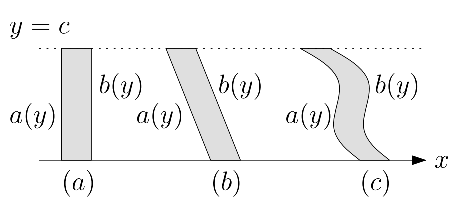


FIGURE 7. Three integration strips with integration boundaries $a(y)$ and $b(y)$.

From Cavalieri's principle it follows that we can easily compute the area of any integration strips.

Proposition 3.9 (Cavalieri's principle for integration strips.)**.** *The area of an integration strip is equal to*

$$A = (b - a)c.$$

Proof. The area of an integration strip can be determined by calculating the area between the curves $b(y)$ and $a(y)$ with the definite integral

$$\begin{aligned} A &= \int_0^c b(y) - a(y)\ dy \\ &= \int_0^c (b-a)\ dy \\ &= (b-a)y\Big|_0^c \\ &= (b-a)c. \end{aligned}$$

□

In order to find the area of a Cavalieri region R, we need to associate two related partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ to the region R.

Definition 3.10. A partition of $[a, b]$ is a finite set $\mathcal{P}$ of points $x_0, x_1, \ldots, x_n$ such that $a = x_0 < x_1 < \cdots < x_n = b$. We describe $\mathcal{P}$ by writing:

$$\mathcal{P} = \{x_0, x_1, \ldots, x_n\}.$$

The n subintervals into which a partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ divides $[a, b]$ are $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$. Their lengths are $x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}$, respectively. We denote the length $x_k - x_{k-1}$ of the kth subinterval by Δx_k. Thus

$$\Delta x_k = x_k - x_{k-1}$$

and we define

$$\Delta x_0 := 0.$$

We now choose any partition $\mathcal{P}_1 = \{x_0^1, x_1^1, \ldots, x_n^1\}$ of $[a, b]$, and we inscribe over each subinterval derived from $\mathcal{P}_1$ the largest integration strip that lies inside the Cavalieri region R. Since both boundaries of any integration strip are necessarily translations of the translational function $a(y)$, we can apply the transformation function h to the partition $\mathcal{P}_1$. If the transformation function h is strictly increasing, the restriction of h to the partition $\mathcal{P}_1$ induces a new partition $\mathcal{P}_2 = \{x_0^2, x_1^2, \ldots, x_n^2\}$ as shown in Figure 8. Otherwise, if h is strictly decreasing, the restriction of h induces a reversed partition $\mathcal{P}_2 = \{x_n^2, x_{n-1}^2, \ldots, x_0^2\}$. In the rest of this document we will assume that h is strictly increasing, without any loss of generality.

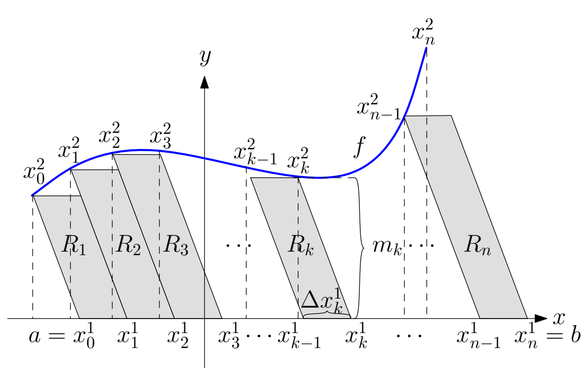


FIGURE 8. The lower Cavalieri sum of $f(x)$ for a partition $\mathcal{P}_1$ on $[a, b]$.

Since we have assumed that f is continuous and nonnegative on $[a', b']$, we know from the Maximum-Minimum theorem that for each k between 1 and n there exists a smallest value m_k of f on the kth subinterval $[x^2_{k-1}, x^2_k]$. If we choose m_k as the height of the kth integration strip R_k, then R_k will be the largest (tallest) integration strip that can be inscribed in R over $[x^1_{k-1}, x^1_k]$. Doing this for each subinterval, we create n inscribed strips $R_1, R_2, \ldots, R_n$, all lying inside the region R. For each k between 1 and n the strip R_k has base $[x^1_{k-1}, x^1_k]$ with width Δx^1_k and has height m_k. Hence the area of R_k is the product $m_k \Delta x^1_k$ (by Cavalieri's principle). The sum

$$L(\mathcal{P}_1, f, h) = \sum_{k=1}^{n} m_k \Delta x^1_k, \qquad \text{(lower Cavalieri sum)}$$

where

$$m_k = \inf_x f(x), \quad h(x^1_{i-1}) = x^2_{i-1} \le x \le x^2_i = h(x^1_i)$$

is called the lower Cavalieri sum and should be no larger than the area of R. The lower Cavalieri sum is represented graphically in Figure 8. Recall that the lower Riemann sum is defined similarly, that is

$$L(\mathcal{P}, f) = \sum_{k=1}^{n} m_k \Delta x_k, \qquad \text{(lower Riemann sum)}$$

where

$$m_k = \inf_x f(x), \quad x_{i-1} \le x \le x_i$$

and $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is a partition on $[a, b]$, and the integration strips are rectangular. The lower Riemann sum is represented graphically in Figure 9.

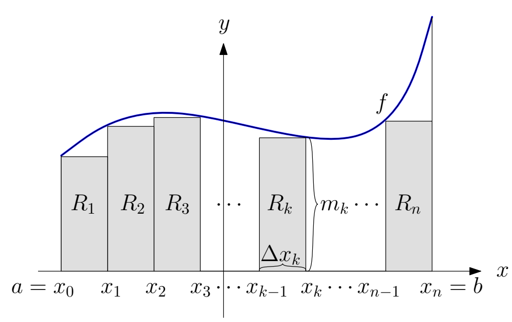


FIGURE 9. The Lower Riemann Sum of $f(x)$ for a partition $\mathcal{P}$ on $[a, b]$.

Irrespective of how we define the area of the Cavalieri region R, this area must be at least as large as the lower Cavalieri sum $L(\mathcal{P}_1, f, h)$ associated with any partition $\mathcal{P}_1$ of $[a, b]$.

By a procedure similar to the one that involves inscribing integration strips to compute a lower Cavalieri sum, we can also circumscribe integration strips and compute an upper Cavalieri sum as shown in Figure 10.

Let $\mathcal{P}_1 = \{x^1_0, x^1_1, \ldots, x^1_n\}$ be a given partition of $[a, b]$, and let f be continuous and nonnegative on $[a', b']$. The Maximum-Minimum Theorem implies that for each k between 1 and n there exists a largest value M_k of f on the kth integration strip

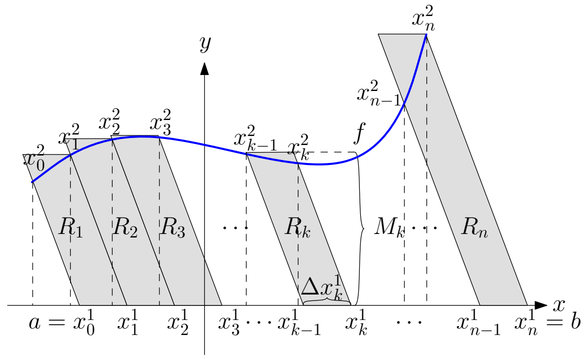


FIGURE 10. The upper Cavalieri sum of $f(x)$ for a partition $\mathcal{P}_1$ on $[a, b]$.

R_k, such that R_k will be the smallest possible strip circumscribing the appropriate portion of R. The area of R_k is $M_k \Delta x_k^1$, and the sum

$$U(\mathcal{P}_1, f, h) = \sum_{k=1}^{n} M_k \Delta x_k^1, \qquad \text{(upper Cavalieri sum)}$$

where

$$M_k = \sup_x f(x), \quad h(x_{i-1}^1) = x_{i-1}^2 \leq x \leq x_i^2 = h(x_i^1)$$

is called the upper Cavalieri sum of f associated with the partition $\mathcal{P}_1$. The upper Cavalieri sum is represented graphically in Figure 10. Recall that the upper Riemann sum is defined similarly, that is

$$U(\mathcal{P}, f) = \sum_{k=1}^{n} M_k \Delta x_k, \qquad \text{(upper Riemann sum)}$$

where

$$M_k = \sup_x f(x), \quad x_{i-1} \leq x \leq x_i$$

and $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is a partition on $[a, b]$, and the integration strips are rectangular. The upper Riemann sum is represented graphically in Figure 11.

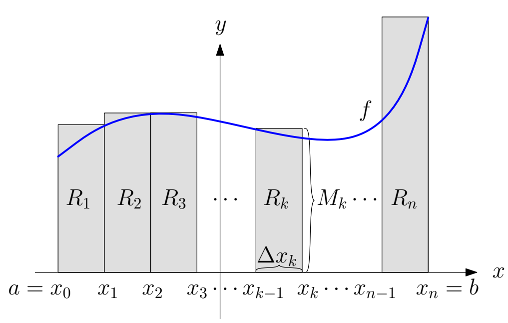


FIGURE 11. The upper Riemann sum of $f(x)$ for a partition $\mathcal{P}$ on $[a, b]$.

Irrespective of how we define the area of the Cavalieri region R, this area must be no larger than the upper Cavalieri sum $U(\mathcal{P}_1, f, h)$ for any partition $\mathcal{P}_1$ of $[a, b]$.

The assumption that f must be nonnegative on $[a', b']$ can now be dropped. Assuming only that f is continuous on $[a', b']$, we still define the lower and upper Cavalieri sums of f for a partition $\mathcal{P}_1$ of $[a, b]$ by

$$L(\mathcal{P}_1, f, h) = \sum_{k=1}^{n} m_k \Delta x_k^1$$

and

$$U(\mathcal{P}_1, f, h) = \sum_{k=1}^{n} M_k \Delta x_k^1,$$

where for any integer k between 1 and n, m_k and M_k are the minimum and maximum values of f on $[x_{k-1}^2, x_k^2]$, respectively.

Remark 3.11. In the rest of this document we will repeatedly make use of the following notation. We will let $f(x)$ be any continuous function on the interval $[a', b']$. We will also assume that $a(y)$ is some translational function w.r.t. $f(x)$ on the interval $[a, b]$, with which we'll associate a partition $\mathcal{P}_1$. Furthermore, we will let h denote the transformation function which maps the partition $\mathcal{P}_1 \subset [a, b]$ to the partition $\mathcal{P}_2 \subset [a', b']$. Of course, $b(y)$ must be a particular translation on the x-axis of $a(y)$, such that $b(y) = a(y) + (b - a)$, where $a = a(0)$ and $b = b(0)$ as defined previously. Finally, we have that a' and b' are the unique x-values for which $a(y)$ and $b(y)$ intersect $f(x)$, respectively.

Definition 3.12 (Cavalieri sum)**.** For each $k \in \mathbb{N}$ from 1 to n, let t_k^2 be an arbitrary number in $[x_{k-1}^2, x_k^2] \subseteq [a', b']$. Then the sum

$$\mathcal{C}(\mathcal{P}_1, f, h) = \sum_{k=1}^{n} f(t_k^2)\Delta x_k^1 = f(t_1^2)\Delta x_1^1 + f(t_2^2)\Delta x_2^1 + \cdots + f(t_n^2)\Delta x_n^1$$

is called a Cavalieri sum for f on $[a, b]$.

Recall that a Riemann sum for f on $[a, b]$ is defined similarly, that is

$$\mathcal{R}(\mathcal{P}, f) = \sum_{k=1}^{n} f(t_k)\Delta x_k = f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \cdots + f(t_n)\Delta x_n,$$

where $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is any partition of $[a, b]$, and t_k is an arbitrary number in $[x_{k-1}, x_k] \subseteq [a, b]$.

Proposition 3.13. *The lower Cavalieri sum $L(\mathcal{P}_1, f, h)$ is equivalent to the lower Riemann sum $L(\mathcal{P}_1, f \circ h)$, that is*

$$L(\mathcal{P}_1, f, h) = L(\mathcal{P}_1, f \circ h) \tag{3.1}$$

and the upper Cavalieri sum $U(\mathcal{P}_1, f, h)$ is equivalent to the upper Riemann sum $U(\mathcal{P}_1, f \circ h)$:

$$U(\mathcal{P}_1, f, h) = U(\mathcal{P}_1, f \circ h). \tag{3.2}$$

Proof. We first consider the lower sums of (3.1). Since the transformation function h is strictly monotone, continuous and bijective on $[a, b]$ we can choose values of x_i^2 to minimize the value of f in the interval $[x_{k-1}^2, x_k^2]$ and so minimizing $f \circ h$ in the interval $[x_{k-1}^1, x_k^1]$. The proof of (3.2) is similar. □

Remark 3.14. Proposition 3.13 will be used repeatedly to prove many of the remaining results for Cavalieri integration, since existing results for Riemann sums will hold trivially for the corresponding Cavalieri sums.

We now give two important results from Riemann integration theory.

Proposition 3.15. *Suppose $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is a partition of the closed interval $[a, b]$, and f a bounded function defined on that interval. Then we have:*

- *The lower Riemann sum is increasing with respect to refinements of partitions, i.e. $L(\mathcal{P}', f) \geq L(\mathcal{P}, f)$ for every refinement $\mathcal{P}'$ of the partition $\mathcal{P}$.*
- *The upper Riemann sum is decreasing with respect to refinements of partitions, i.e. $U(\mathcal{P}', f) \leq U(\mathcal{P}, f)$ for every refinement $\mathcal{P}'$ of the partition $\mathcal{P}$.*
- *$L(\mathcal{P}, f) \leq \mathcal{R}(\mathcal{P}, f) \leq U(\mathcal{P}, f)$ for every partition $\mathcal{P}$.*

Proof. The proof is taken from [6]. The last statement is simple to prove: take any partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$. Then $\inf\{f(x), x_{k-1} \leq x \leq x_k\} \leq f(t_k) \leq \sup\{f(x), x_{k-1} \leq x \leq x_k\}$ where t_k is an arbitrary number in $[x_{k-1}, x_k]$ and $k = 1, 2, \ldots, n$. That immediately implies that $L(\mathcal{P}, f) \leq \mathcal{R}(\mathcal{P}, f) \leq U(\mathcal{P}, f)$. The other statements are somewhat trickier. In the case that one additional point t_0 is added to a particular subinterval $[x_{k-1}, x_k]$, let $c_k = \sup f(x)$ in the interval $[x_{k-1}, x_k]$, $A_k = \sup f(x)$ in the interval $[x_{k-1}, t_0]$, $B_k = \sup f(x)$ in the interval $[x_0, x_k]$. Then $c_k \geq A_k$ and $c_k \geq B_k$ so that:

$$\begin{aligned} c_k(x_k - x_{k-1}) &= c_k(x_k - t_0 + t_0 - x_{k-1}) \\ &= c_k(x_k - t_0) + c_k(t_0 - x_{k-1}) \\ &\geq B_k(x_k - t_0) + A_k(x_0 - t_{k-1}), \end{aligned}$$

which shows that if $\mathcal{P} = \{x_0, x_1, \ldots, x_k, x_{k-1}, \ldots, x_n\}$ and $\mathcal{P}' = \{x_0, x_1, \ldots, x_k, t_0, x_{k-1}, \ldots, x_n\}$ then $U(\mathcal{P}', f) \leq U(\mathcal{P}, f)$. The proof for a general refinement $\mathcal{P}'$ of $\mathcal{P}$ uses the same idea plus an elaborate indexing scheme. No more details should be necessary. The proof for the statement regarding the lower sum is analogous. □

Proposition 3.16. *Let f be continuous on $[a, b]$. Then there is a unique number I satisfying*

$$L(\mathcal{P}, f) \leq I \leq U(\mathcal{P}, f)$$

for every partition $\mathcal{P}$ of $[a, b]$.

Proof. The proof is taken from [2]. From Proposition 3.15 it follows that every lower sum of f on $[a, b]$ is less than or equal to every upper sum. Thus the collection $\mathcal{L}$ of all lower sums is bounded above (by an upper sum) and the collection $\mathcal{U}$ of all upper sums is bounded below (by any lower sum). By the Least Upper Bound Axiom, $\mathcal{L}$ has a least upper bound L and $\mathcal{U}$ has a greatest lower bound G. From our preceding remarks it follows that

$$L(\mathcal{P}, f) \leq L \leq G \leq U(\mathcal{P}, f)$$

for each partition $\mathcal{P}$ of $[a, b]$. Moreover, any number I satisfying

$$L(\mathcal{P}, f) \leq I \leq U(\mathcal{P}, f)$$

for each partition $\mathcal{P}$ of $[a, b]$ must satisfy

$$L \leq I \leq G$$

since L is the least upper bound of the lower sums and G is the greatest lower bound of the upper sums. Hence to complete the proof of the theorem it is enough to prove that $L = G$. Let $\epsilon > 0$. Since f is continuous on $[a, b]$, it follows that f is uniformly continuous on $[a, b]$. Thus there is a $\delta > 0$ such that if x and y are in $[a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$. Let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ such that $\Delta x_k < \delta$ for $1 \leq k \leq n$, and let M_k and m_k be, respectively, the largest and smallest values of f on $[x_{k-1}, x_k]$. Then

$$\begin{aligned} U(\mathcal{P}, f) - L(\mathcal{P}, f) &= \sum_{k=1}^{n} M_k \Delta x_k - \sum_{k=1}^{n} m_k \Delta x_k \\ &= \sum_{k=1}^{n} (M_k - m_k) \Delta x_k \\ &< \frac{\epsilon}{b-a} \sum_{k=1}^{n} \Delta x_k \\ &= \frac{\epsilon}{b-a} (b-a) \\ &= \epsilon. \end{aligned}$$

Since $L(\mathcal{P}, f) \leq L \leq G \leq U(\mathcal{P}, f)$, it follows that $0 \leq G - L \leq U(\mathcal{P}, f) - L(\mathcal{P}, f) \leq \epsilon$. Since ϵ was arbitrary, we conclude that $L = G$. □

Definition 3.17 (Definite Riemann integral)**.** Let f be continuous on $[a, b]$. The definite Riemann integral of f from a to b is the unique number I satisfying

$$L(\mathcal{P}, f) \leq I \leq U(\mathcal{P}, f)$$

for every partition $\mathcal{P}$ of $[a, b]$. This integral is denoted by

$$\int_a^b f(x)\ dx.$$

We now state (and prove) the equivalent of Proposition 3.15 for lower and upper Cavalieri sums:

Proposition 3.18. *We clearly have:*

- *The lower Cavalieri sum is increasing with respect to refinements of partitions, i.e.* $L(\mathcal{P}_1', f, h) \geq L_(\mathcal{P}_1, f, h)$ *for every refinement* $\mathcal{P}_1'$ *of the partition* $\mathcal{P}_1$*.*
- *The upper Cavalieri sum is decreasing with respect to refinements of partitions, i.e.* $U(\mathcal{P}_1', f, h) \leq U_(\mathcal{P}_1, f, h)$ *for every refinement* $\mathcal{P}_1'$ *of the partition* $\mathcal{P}_1$*.*
- $L(\mathcal{P}_1, f, h) \leq \mathcal{C}(\mathcal{P}_1, f, h) \leq U(\mathcal{P}_1, f, h)$ *for every partition* $\mathcal{P}_1$*.*

Proof. The proof follows trivially from Proposition 3.13 and Proposition 3.15 (since every Cavalieri sum corresponds to an equivalent Riemann sum). □

Proposition 3.19. *Let f be continuous on $[a', b']$. Then there is a unique number I satisfying*

$$L(\mathcal{P}_1, f, h) \leq I \leq U(\mathcal{P}_1, f, h)$$

for every partition $\mathcal{P}_1$ of $[a, b]$.

Proof. The proof follows trivially from Proposition 3.13 and Proposition 3.16. □

We can now *finally* define the Cavalieri integral:

Definition 3.20 (Definite Cavalieri integral)**.** Let f be continuous on $[a', b']$. The definite Cavalieri integral of $f(x)$ from $a(y)$ to $b(y)$ is the unique number I satisfying

$$L_(\mathcal{P}_1, f, h) \leq I \leq U(\mathcal{P}_1, f, h)$$

for every partition $\mathcal{P}_1$ of $[a, b]$. This integral is denoted by

$$\int_{a(y)}^{b(y)} f(x)\, dx.$$

Definition 3.21. Let R be any Cavalieri region as given in Definition 3.2 then the area A of the region R is defined to be

$$A = \int_{a(y)}^{b(y)} f(x)\, dx.$$

Proposition 3.22. *The following Cavalieri and Riemann integrals are equivalent:*

$$\int_{a(y)}^{b(y)} f(x)\, dx = \int_{a}^{b} f \circ h(x)\, dx.$$

Proof. By noting that $L(\mathcal{P}_1, f \circ h) = L(\mathcal{P}_1, f, h) \leq I \leq U(\mathcal{P}_1, f, h) = U(\mathcal{P}_1, f \circ h)$, the proof follows trivially from Proposition 3.13, Proposition 3.16 and Proposition 3.19. □

Theorem 3.23. *For any $\epsilon > 0$ there is a number $\delta > 0$ such that the following statement holds: If any subinterval of $\mathcal{P}_1$ has length less than δ, and if $x_{k-1}^2 \leq t_k^2 \leq x_k^2$ for each k between 1 and n, then the associated Cavalieri sum $\sum_{k=1}^{n} f(t_k^2)\Delta x_k^1$ satisfies.*

$$\left| \int_{a(y)}^{b(y)} f(x)\ dx - \sum_{k=1}^{n} f(t_k^2)\Delta x_k^1 \right| < \epsilon.$$

Proof. This proof was adapted from [2]. For any $\epsilon > 0$ choose $\delta > 0$ such that if x and y are in $[a', b']$ then $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{b'-a'}$. If $\mathcal{P}_1$ is chosen so that $\Delta x_k^1 < \delta$ for each k, then by Proposition 3.19,

$$U(\mathcal{P}_1, f, h) - L(\mathcal{P}_1, f, h) \leq \epsilon.$$

Moreover, if $x_{k-1}^2 \leq t_k^2 \leq x_k^2$ for $1 \leq k \leq n$, then

$$m_k \leq f(t_k^2) \leq M_k.$$

It follows that

$$L(\mathcal{P}_1, f, h) = \sum_{k=1}^{n} m_k \Delta x_k^1 \leq \sum_{k=1}^{n} f(t_k^2)\Delta x_k^1 \leq \sum_{k=1}^{n} M_k \Delta x_k^1 = U(\mathcal{P}_1, f, h).$$

Since

$$L(\mathcal{P}_1, f, h) \leq \int_{a(y)}^{b(y)} f(x)\, dx \leq U(\mathcal{P}_1, f, h),$$

we conclude that

$$\left| \int_{a(y)}^{b(y)} f(x)\, dx - \sum_{k=1}^{n} f(t_k^2)\Delta x_k^1 \right| < \epsilon.$$

□

By combining Proposition 3.22 and Theorem 3.23 we finally have

$$\begin{aligned}\int_{a(y)}^{b(y)} f(x)\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(t_k^2)\Delta x_k^1 \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} f\circ h(t_k^1)\Delta x_k^1 \\ &= \int_a^b f\circ h(x)\,dx,\end{aligned}$$

where the last line follows from the well known fact that the limit of a Riemann sum equals the Riemann integral.

3.3. **The Cavalieri integral as a Riemann-Stieltjes integral.** When evaluating a Cavalieri integral from $a(y)$ to $b(y)$, it may sometimes be more convenient to consider an equivalent Riemann-Stieltjes integral from a' to b' than the ordinary Riemann integral from a to b.

To transform the Cavalieri integral into an equivalent Riemann-Stieltjes integral, we will make use of the inverse transformation function $g := h^{-1}$ (which is guaranteed to exist, since h is a bijective function).

Definition 3.24 (Inverse transformation function g)**.** Let $a(y)$ be a translational function. The mapping $g : [a', b'] \to [a, b]$, which maps $x_i^2 \in [a', b']$ to $x_i^1 \in [a, b]$, is defined as $g(x_i^2) := x_i^2 - a\circ f(x_i^2) + a$, which we will refer to as the inverse transformation function.

Proposition 3.25. *The following Cavalieri and Riemann-Stieltjes integrals are equivalent:*

$$\int_{a(y)}^{b(y)} f(x)\,dx = \int_{a'}^{b'} f(x)\,dg(x).$$

Proof. From Theorem 3.23 we have

$$\int_{a(y)}^{b(y)} f(x)\,dx = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i^2)\Delta x_i^1. \tag{3.3}$$

By noting that $\Delta x_i^1 = x_{i+1}^1 - x_i^1 = g(x_{i+1}^2) - g(x_i^2)$, and that $g(a') = a$ and $g(b') = b$, we can re-write (3.3) as

$$\int_{a(y)}^{b(y)} f(x)\,dx = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i^2)\big[g(x_{i+1}^2) - g(x_i^2)\big],$$

which we recognize as the Riemann-Stieltjes integral $\int_{a'}^{b'} f(x)\,dg(x)$, as required. □

Whenever g is differentiable, we can conveniently express the Cavalieri integral simply in terms of $f(x)$ and $a(y)$:

$$\int_{a(y)}^{b(y)} f(x)\,dx = \int_{a'}^{b'} f(x)\left[1 - \frac{da(y)}{dy}\circ f(x)\cdot\frac{df(x)}{dx}\right]dx.$$

4. Cavalieri Integration: Worked Examples

Several fully worked examples of Cavalieri integration are given below. We first present a simple example of Cavalieri integration from first principles (Example 4.1), followed by the integration of a Cavalieri region in which $f(x)$ is nonlinear (Example 4.2). In Example 4.3 the boundary functions are also nonlinear, followed by Example 4.4 in which the boundary function $b(y)$ is no longer required to be a translation of $a(y)$. In Example 4.5 we show that the transformation function h can be fiendishly difficult to find, but we show that the Riemann-Stieltjes formulation leads to a much simpler solution in Example 4.6. In Example 4.7 we show that the Cavalieri integral can be used to integrate non-Cavalieri regions (with $a(y)$ non-translational), and in Example 4.8 we show that the transformation function h can be strictly decreasing. Finally, in Example 4.9 we show that the Cavalieri integral can be used in some instances where the function $f(x)$ is not even defined.

Example 4.1 (Cavalieri integration from first principles; $f(x)$, $a(y)$ and $b(y)$ linear)**.** Consider the Cavalieri region bounded by the x-axis and the lines $f(x) = x$, $a(y) = 1 - y$, and $b(y) = 4 - y$. This region is shown in Figure 12.

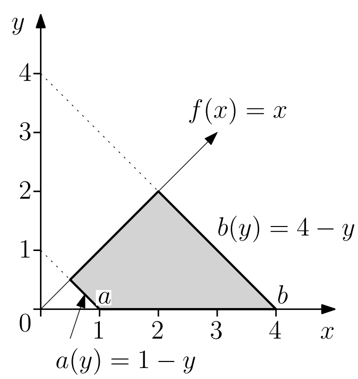


Figure 12. Region bounded by the x-axis and the lines $f(x) = x$, $a(y) = 1 - y$, and $b(y) = 4 - y$.

Also consider a partition $(x_i^1)_{i=0}^n$ on the x-axis such that $a = x_0^1 < x_1^1 < \cdots < x_n^1 = b$, and $\Delta x_i^1 = x_{i+1}^1 - x_i^1$. We can form the Cavalieri integral (using the left hand rule) as follows:

$$\int_{a(y)}^{b(y)} f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i^2)\Delta x_i^1. \tag{4.1}$$

The partition points x_i^2 as used in the Cavalieri sum is shown in Figure 13.

To transform the Cavalieri sum given in (4.1) into an ordinary Riemann sum, we must find an expression for x_i^2 in terms of the partition points x_i^1, for all $i = 0, 1, \ldots, n$. First consider the collection of functions $\{a(y) + [x_i^1 - a] = x_i^2 : i = 0, 1, \ldots, n\}$. To find the partition points x_i^2 in terms of x_i^1 we substitute the function

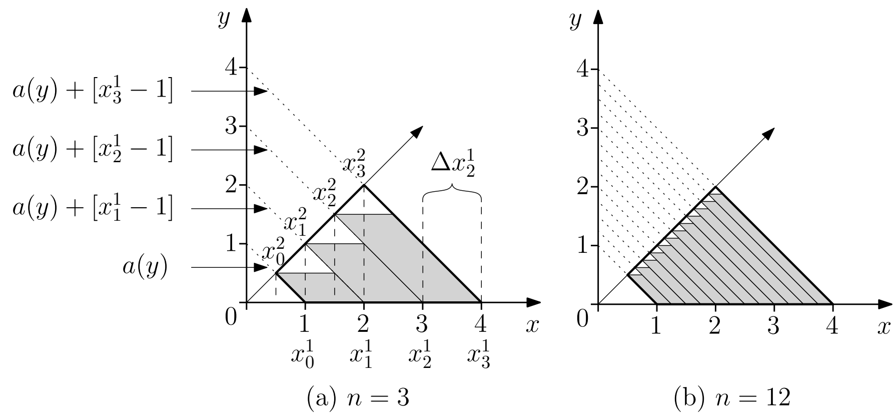


FIGURE 13. Partition points x_i^2 as used in the Cavalieri sum.

$f(x_i^2)$ for y to obtain:

$$\begin{aligned} a\circ f(x_i^2)+[x_i^1-1] &= x_i^2 \\ -x_i^2+x_i^1 &= x_i^2 \\ x_i^2 &= \frac{x_i^1}{2}, \end{aligned}$$

so that we have the general expression $x_i^2=h(x_i^1)$, with $h(x)=x/2$.

Finally this allows us to rewrite the Cavalieri integral from (4.1) as an equivalent Riemann integral:

$$\begin{aligned} \int_{a(y)}^{b(y)} f(x)\,dx &= \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i^2)\Delta x_i^1 \\ &= \lim_{n\to\infty}\sum_{i=0}^{n-1} f\circ h(x_i^1)\Delta x_i^1 \\ &= \int_a^b f\circ h(x)\,dx. \end{aligned} \tag{4.2}$$

Evaluating the Riemann integral of (4.2) with $a=1$ and $b=4$ we obtain

$$\begin{aligned} \int_{a(y)}^{b(y)} f(x)\,dx &= \int_a^b f\circ h(x)\,dx \\ &= \frac{1}{2}\int_1^4 x\,dx \\ &= \frac{1}{4}x^2\Big|_1^4 \\ &= 3.75, \end{aligned}$$

which we can quickly verify to be correct by evaluating the area of the region shown in Figure 12 with ordinary Riemann integration:

$$\begin{aligned}\int_0^2 x\,dx + \int_2^4 4 - x\,dx - \int_0^{\frac{1}{2}} x\,dx - \int_{\frac{1}{2}}^1 1 - x\,dx = 3.75\\ = \int_{a(y)}^{b(y)} f(x)\,dx.\end{aligned}$$

Example 4.2 (Cavalieri integration; $f(x)$ nonlinear)**.** Consider the Cavalieri region bounded by the x-axis and the functions $f(x) = x^2$, $a(y) = 1 - y$, and $b(y) = 4 - y$. This region is shown in Figure 14, along with the strips of integration.

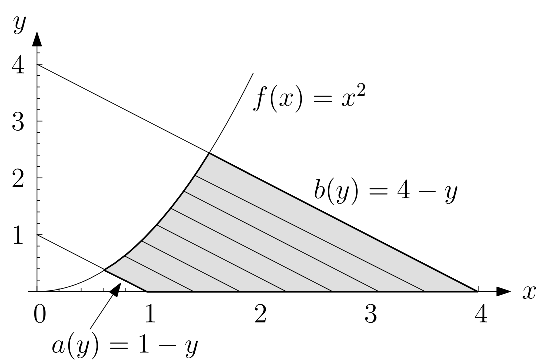


FIGURE 14. Region bounded by the x-axis and the functions $f(x) = x^2$, $a(y) = 1 - y$, and $b(y) = 4 - y$.

The area of this region can be calculated with the Cavalieri integral

$$\int_{a(y)}^{b(y)} f(x)\,dx = \int_a^b f \circ h(x)\,dx. \tag{4.3}$$

To evaluate (4.3) we first need to find h using Definition 3.3:

$$\begin{aligned} a \circ f(x_i^2) + [x_i^1 - 1] &= x_i^2 \\ -(x_i^2)^2 + 1 + x_i^1 - 1 &= x_i^2 \\ (x_i^2)^2 + x_i^2 - x_i^1 &= 0 \\ x_i^2 &= \frac{1}{2}\left(\sqrt{4x_i^1 + 1} - 1\right) \\ &= h(x_i^1). \end{aligned}$$

We can now calculate (4.3) with $h(x) = \frac{1}{2}(\sqrt{4x+1} - 1)$ as follows

$$\begin{aligned}\int_{a(y)}^{b(y)} f(x)\,dx &= \int_a^b f \circ h(x)\,dx \\ &= \frac{1}{4}\int_1^4 (\sqrt{4x+1}-1)^2\,dx \\ &= -\frac{1}{96}(4x+1)(-12x+8\sqrt{4x+1}-9)\Big|_1^4 \\ &= 9 + \frac{1}{12}(5\sqrt{5} - 17\sqrt{17}) \\ &\approx 4.09063.\end{aligned}$$

One can also compute the area under consideration (see Figure 14) using ordinary Riemann integration:

$$\begin{aligned}&\int_0^{\frac{1}{2}(\sqrt{17}-1)} x^2\,dx + \int_{\frac{1}{2}(\sqrt{17}-1)}^4 4 - x\,dx \\ &- \int_0^{\frac{1}{2}(\sqrt{5}-1)} x^2\,dx - \int_{\frac{1}{2}(\sqrt{5}-1)}^1 1 - x\,dx \approx 4.09063 \\ &\qquad \approx \int_{a(y)}^{b(y)} f(x)\,dx.\end{aligned}$$

Example 4.3 (Cavalieri integration; $f(x)$, $a(y)$ and $b(y)$ nonlinear)**.** Consider the Cavalieri region bounded by the x-axis and the functions $f(x) = x^2$, $a(y) = 2-\sqrt{y}$, and $b(y) = 4 - \sqrt{y}$. This region is shown in Figure 15, along with the strips of integration.

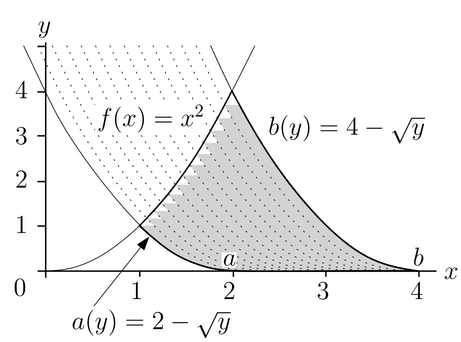


FIGURE 15. Region bounded by the x-axis and the functions $f(x) = x^2$, $a(y) = 2 - \sqrt{y}$, and $b(y) = 4 - \sqrt{y}$.

The area of this region can be calculated with the Cavalieri integral

$$\int_{a(y)}^{b(y)} f(x)\,dx = \int_a^b f \circ h(x)\,dx. \tag{4.4}$$

To evaluate (4.4) we first need to find h:

$$\begin{aligned} a \circ f(x_i^2) + [x_i^1 - 2] &= x_i^2 \\ -x_i^2 + 2 + x_i^1 - 2 &= x_i^2 \\ x_i^2 &= \frac{1}{2} x_i^1 \\ &= h(x_i^1). \end{aligned}$$

We can now calculate (4.4) with $h(x) = x/2$ as follows

$$\begin{aligned} \int_{a(y)}^{b(y)} f(x)\, dx &= \int_a^b f \circ h(x)\, dx \\ &= \frac{1}{4} \int_2^4 x^2\, dx \\ &= \frac{1}{12} x^3 \Big|_2^4 \\ &= \frac{14}{3}. \end{aligned}$$

We can once again verify our answer above by computing the area of the region shown in Figure 15 with ordinary Riemann integration:

$$\begin{aligned} \int_0^2 x^2\, dx + \int_2^4 (4-x)^2\, dx - \int_0^1 x^2\, dx - \int_1^2 (2-x)^2\, dx &= \frac{14}{3} \\ &= \int_{a(y)}^{b(y)} f(x) dx. \end{aligned}$$

Example 4.4 (Cavalieri integration; $b(y)$ not a translation of $a(y)$)**.** Consider the Cavalieri region bounded by the x-axis and the functions $f(x) = \sqrt{x}$, $a(y) = 2 - y^2$, and $b(y) = 4 - y$. This region is shown in Figure 16.

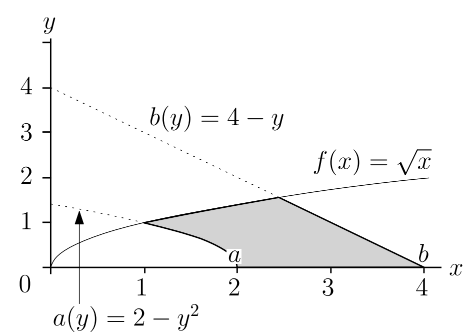


FIGURE 16. Region bounded by the x-axis and the functions $f(x) = \sqrt{x}$, $a(y) = 2 - y^2$, and $b(y) = 4 - y$.

To obtain the shaded area bounded in Figure 16 we will subtract the two Cavalieri integrals shown in Figure 17 and Figure 18. That is, we will compute the desired area by evaluating $A - B$.

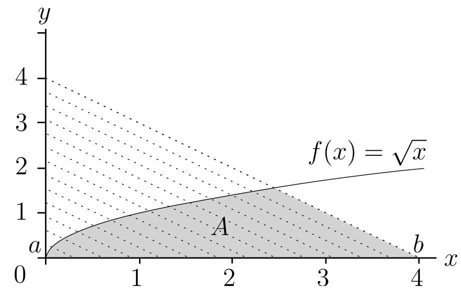


FIGURE 17. Region bounded by the x-axis and the functions $f(x) = \sqrt{x}$ and $b(y) = 4 - y$.

The area of A can be calculated with the Cavalieri integral

$$\int_{-y}^{b(y)} f(x)\,dx = \int_0^b f \circ h_1(x)\,dx. \tag{4.5}$$

To evaluate (4.5) we first need to find h_1:

$$\begin{aligned}
-f(x_i^2) + x_i^1 &= x_i^2 \\
-\sqrt{x_i^2} + x_i^1 &= x_i^2 \\
\sqrt{x_i^2} + x_i^2 - x_i^1 &= 0 \\
x_i^2 &= \left(x_i^1 + \frac{1}{2}\right) - \frac{1}{2}\sqrt{4x_i^1 + 1} \\
&= h_1(x_i^1).
\end{aligned}$$

We can now calculate (4.5) with $h_1(x) = (x + 0.5) - 0.5\sqrt{4x+1}$ as follows:

$$\begin{aligned}
\int_{-y}^{b(y)} f(x)\,dx &= \int_0^b f \circ h_1(x)\,dx \\
&= \int_0^4 \sqrt{(x + 0.5) - 0.5\sqrt{4x+1}}\,dx \\
&= \frac{1}{12}(4x+1)^{\frac{3}{2}} - \frac{1}{2}x - \frac{1}{8}\Big|_0^4 \\
&\approx 3.75773.
\end{aligned}$$

The area of B can be calculated with the Cavalieri integral (see Figure 18)

$$\int_{-y^2}^{a(y)} f(x)\,dx = \int_0^a f \circ h_2(x)\,dx. \tag{4.6}$$

To evaluate (4.6) we first need to find h_2:

$$\begin{aligned}
-(f(x_i^2))^2 + x_i^1 &= x_i^2 \\
-x_i^2 + x_i^1 &= x_i^2 \\
x_i^2 &= \frac{1}{2}x_i^1
\end{aligned}$$

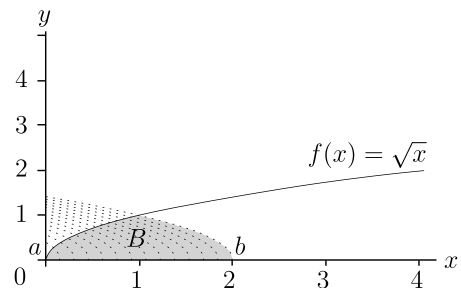


FIGURE 18. Region bounded by the x-axis and the functions $f(x) = \sqrt{x}$ and $a(y) = 2 - y^2$.

We can now calculate (4.6) with $h_2(x) = x/2$ as

$$\begin{aligned}\int_{-y^2}^{a(y)} f(x)\,dx &= \int_0^a f \circ h_2(x)\,dx \\ &= \int_0^2 \sqrt{0.5x}\,dx \\ &= \left.\frac{2x^{\frac{3}{2}}}{3\sqrt{2}}\right|_0^2 \\ &= \frac{4}{3}.\end{aligned}$$

Finally we obtain the desired area $A - B = \int_{-y}^{b(y)} f(x)\,dx - \int_{-y^2}^{a(y)} f(x)\,dx \approx 3.75773 - \dfrac{4}{3} \approx 2.42440$.

Example 4.5 (Cavalieri integration; h difficult to find)**.** Consider the Cavalieri region bounded by the x-axis and the functions $f(x) = x^2$, $a(y) = 1 - y^2$, and $b(y) = 4 - y^2$. This region is shown in Figure 19, along with the strips of integration.

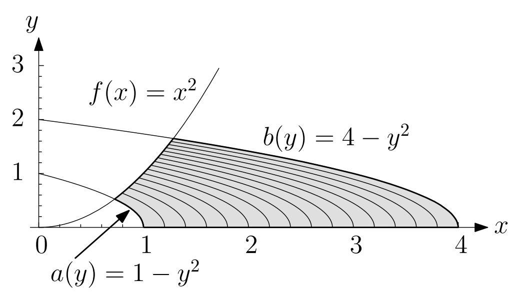


FIGURE 19. Region bounded by the x-axis and the functions $f(x) = x^2$, $a(y) = 1 - y^2$, and $b(y) = 4 - y^2$.

The area of this region can be calculated with the Cavalieri integral

$$\int_{a(y)}^{b(y)} f(x)\,dx = \int_a^b f \circ h(x)\,dx. \tag{4.7}$$

To evaluate (4.7) we first need to find h:

$$\begin{aligned} a \circ f(x_i^2) + [x_i^1 - 1] &= x_i^2 \\ -(x_i^2)^4 + 1 + x_i^1 - 2 &= x_i^2 \\ (x_i^2)^4 + x_i^2 - x_i^1 &= 0. \end{aligned}$$

Solving for x_i^2 in terms of x_i^1 produces $h(x)$ which is equal to:

$$h(x) = \frac{1}{2}\sqrt{\frac{2}{\sqrt{G(x)}} - G(x)} - \frac{1}{2}\sqrt{G(x)} \tag{4.8}$$

with

$$G(x) = \frac{\sqrt[3]{\sqrt{3}.\sqrt{256x^3+27}+9}}{\sqrt[3]{2}.3^{\frac{2}{3}}} - \frac{4\sqrt[3]{\frac{2}{3}}x}{\sqrt[3]{\sqrt{3}.\sqrt{256x^3+27}+9}}.$$

We can now calculate (4.7) with $h(x)$ given by (4.8) as follows:

$$\begin{aligned} \int_{a(y)}^{b(y)} f(x)\,dx &= \int_a^b f \circ h(x)\,dx \\ &= \int_1^4 \left(\frac{1}{2}\sqrt{\frac{2}{\sqrt{G(x)}} - G(x)} - \frac{1}{2}\sqrt{G(x)}\right)^2 dx \\ &\approx 3.46649, \end{aligned}$$

which we will once again verify by using ordinary Riemann integration:

$$\begin{aligned} &\int_0^{1.28378} x^2\,dx + \int_{1.28378}^4 \sqrt{4-x}\,dx \\ &- \int_0^{0.724492} x^2\,dx - \int_{0.724492}^1 \sqrt{1-x}\,dx \approx 3.46649 \\ &\qquad\qquad \approx \int_{a(y)}^{b(y)} f(x)\,dx. \end{aligned}$$

Example 4.6 (Riemann-Stieltjes formulation)**.** Consider the Cavalieri region bounded by the x-axis and the functions $f(x) = x^2$, $a(y) = 1 - y$, and $b(y) = 4 - y$. This region is shown in Figure 20, along with the strips of integration. Note that this is the same region as studied in Example 4.2. We will show that the Riemann-Stieltjes formulation is considerably simpler than the direct method in which we need to find h explicitly.

The area of this region can be calculated with the Cavalieri integral

$$\int_{a(y)}^{b(y)} f(x)\,dx = \int_{a'}^{b'} f(x)\,dg(x). \tag{4.9}$$

To evaluate (4.9) we first need to find g using Definition 3.24:

$$\begin{aligned} x_i^1 &= x_i^2 - a \circ f(x_i^2) + 1 \\ &= (x_i^2)^2 + x_i^2 \\ &= g(x_i^2). \end{aligned}$$

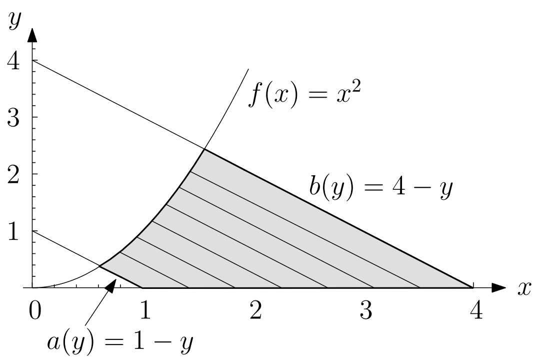


FIGURE 20. Region bounded by the x-axis and the functions $f(x) = x^2$, $a(y) = 1 - y$, and $b(y) = 4 - y$.

We can now calculate (4.9) with $g(x) = x^2 + x$ as follows

$$\begin{aligned}\int_{a(y)}^{b(y)} f(x)\,dx &= \int_{a'}^{b'} f(x)\,dg(x)\\ &= \int_{a'}^{b'} f(x)g'(x)\,dx\\ &= \int_{\frac{1}{2}(\sqrt{5}-1)}^{\frac{1}{2}(\sqrt{17}-1)} x^2(2x+1)\,dx\\ &= \frac{x^4}{2} + \frac{x^3}{3}\Bigg|_{\frac{1}{2}(\sqrt{5}-1)}^{\frac{1}{2}(\sqrt{17}-1)}\\ &\approx 4.09063,\end{aligned}$$

which is the same as obtained in Example 4.2.

Example 4.7 (Cavalieri integration; $a(y)$ non-translational)**.** Consider the *non-Cavalieri* region R shown in Figure 21.A:

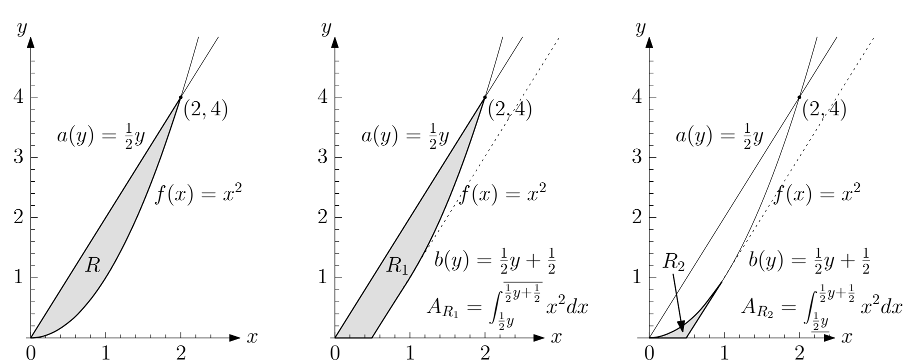


FIGURE 21. The region bounded by $f(x) = x^2$ and $a(y) = \frac{1}{2}y$.

The area of this region can be calculated with the double integral:

$$\begin{aligned} A_R &= \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} 1\, dx\, dy \\ &= \int_0^4 x\Big|_{\frac{1}{2}y}^{\sqrt{y}} dy \\ &= \int_0^4 \sqrt{y} - \frac{1}{2}y\, dy \\ &= \frac{2}{3}y^{\frac{3}{2}} - \frac{1}{4}y^2\Big|_0^4 \\ &= \frac{4}{3}, \end{aligned}$$

and also with the integral:

$$\begin{aligned} A_R &= \int_0^2 2x - x^2\, dx \\ &= x^2 - \frac{1}{3}x^3\Big|_0^2 \\ &= \frac{4}{3}. \end{aligned}$$

We can also calculate the area A_R with the difference between two Cavalieri integrals. The two areas being subtracted are shown in Figure 21.B and Figure 21.C.

$$\begin{aligned} A_R &= A_{R_1} - A_{R_2} \\ &= \int_{\frac{1}{2}y}^{\overline{\frac{1}{2}y+\frac{1}{2}}} x^2 - \int_{\underline{\frac{1}{2}y}}^{\frac{1}{2}y+\frac{1}{2}} x^2 \\ &= \int_0^{\frac{1}{2}} f\circ h_1(x)\, dx - \int_0^{\frac{1}{2}} f\circ h_2(x)\, dx \\ &= \int_0^{\frac{1}{2}} \left(1+\sqrt{1-2x}\right)^2 dx - \int_0^{\frac{1}{2}} \left(1-\sqrt{1-2x}\right)^2 dx \\ &= \frac{4}{3}. \end{aligned}$$

Example 4.8 (Cavalieri integration; h strictly decreasing)**.** Consider the Cavalieri region bounded by the x-axis and the functions $f(x) = 3 - 2x$, $a(y) = 2 - y$, and $b(y) = 3 - y$. This region is shown in Figure 22.

The area of this region can be calculated with the Cavalieri integral

$$\int_{a(y)}^{b(y)} f(x)\, dx = \int_a^b f\circ h(x)\, dx. \tag{4.10}$$

To evaluate (4.10) we first need to find h:

$$\begin{aligned} a\circ f(x_i^2) + [x_i^1 - 2] &= x_i^2 \\ \Rightarrow x_i^2 &= 3 - x_i^1, \end{aligned}$$

so that $h(x) = 3 - x$.

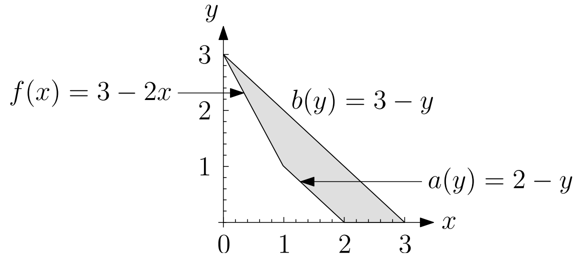


FIGURE 22. Region bounded by the x-axis and the functions $f(x) = 3 - 2x$, $a(y) = 2 - y$, and $b(y) = 3 - y$.

We can now calculate (4.10) as follows

$$\begin{aligned}\int_{a(y)}^{b(y)} f(x)\,dx &= \int_a^b f \circ h(x)\,dx\\ &= \int_2^3 2x - 3\,dx\\ &= x^2\Big|_2^3 - 3x\Big|_2^3\\ &= 2.\end{aligned}$$

Example 4.9 (Cavalieri integration; $f(x)$ not defined)**.** Consider the normal Riemann integration task given below (shown in Figure 23):

$$\int_0^1 \sqrt{1-x}\,dx - \int_0^{0.5} \sqrt{0.5-x}\,dx = \int_0^1 1 - y^2\,dy - \int_0^{\sqrt{0.5}} 0.5 - y^2\,dy \approx 0.431.$$

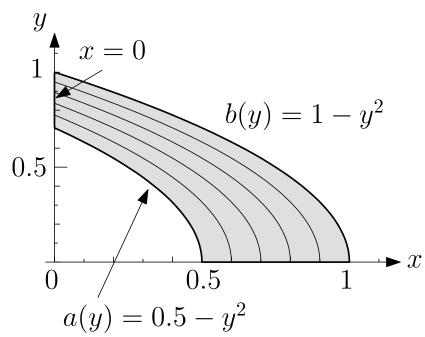


FIGURE 23. Region bounded by the x-axis, the line $x = 0$ and the functions $a(y) = 0.5 - y^2$ and $b(y) = 1 - y^2$.

Note that the shaded region in Figure 23 is not a Cavalieri region, since $f(x)$ is not even defined. Nevertheless, we can compute this area as a single Cavalieri integral as follows. We find the transformation function h by equating

$$a(x_i^2) + [x_i^1 - 0.5] = x_i^2\big|_{x_i^2=0},$$

where the x_i^2 on the right hand side is set to zero since the region is bounded from the left by $x = 0$, and the x_i^2 on the left hand side remains unchanged, since we are really interested in the y-intercepts of each translation of $a(y)$. Therefore we find

$$x_i^2 = h(x_i^1) = \sqrt{x_i^1},$$

so that we can compute the shaded area as

$$\begin{aligned}\int_{a(y)}^{b(y)} (x=0)\,dx &= \int_a^b h(x)\,dx \\ &= \int_{0.5}^{1} \sqrt{x}\,dx \\ &\approx 0.431.\end{aligned}$$

5. Conclusion

We have presented a novel integral $\int_{a(y)}^{b(y)} f(x)\,dx$ in which non-rectangular integration strips were used. We also presented two methods of evaluating Cavalieri integrals by establishing the following relationships between Cavalieri, Riemann and Riemann-Stieltjes integrals:

$$\int_{a(y)}^{b(y)} f(x)\,dx = \int_a^b f \circ h(x)\,dx = \int_{a'}^{b'} f(x)\,dg(x),$$

which is equivalent to noting that

$$\text{Area } A = \text{Area } B = \text{Area } C,$$

as shown in Figure 24.

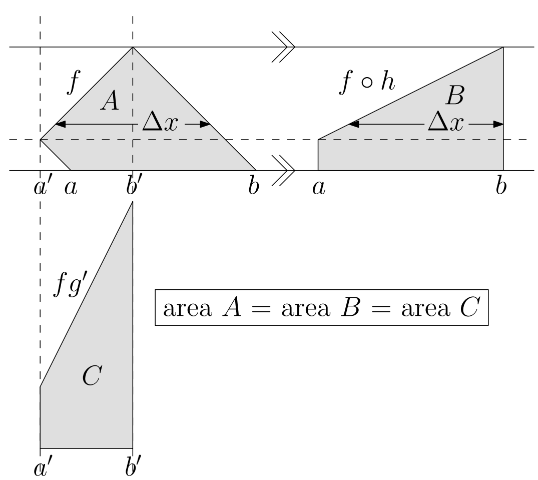


Figure 24. Relationships between Cavalieri (region A), Riemann (region B) and Riemann-Stieltjes (region C) integrals.

The reason for calling $\int_{a(y)}^{b(y)} f(x)\,dx$ the *Cavalieri integral* should now become transparently clear: the area of region B is equal to the area of region A by Cavalieri's principle.

References

[1] K. Andersen, "Cavalieri's Method of Indivisibles," *Arch. Hist. Exact Sci.,* vol. 31, no. 4, pp. 291–367, 1985.

[2] R. Ellis and D. Gulick, *Calculus with analytic geometry.* (4th ed.), Harcourt Brace Jovanovich College Publisher, Orlando, Florida, 1990, pp. A-12–A-13.

[3] H. Eves, "Two Surprising Theorems on Cavalieri Congruence," *College Math. J.,* vol. 22, no. 2, pp. 118–124, 1991.

[4] J. J. O' Connor and E. F. Robertson, *Antiphon the Sophist*, 1999. [Online]. Available: http://www-history.mcs.st-andrews.ac.uk/Biographies/Antiphon.html

[5] A. A. Ruffa, "The generalized method of exhaustion," *Int. J. Math. Math. Sci.,* vol. 31, no. 6, pp. 345–351, 2002.

[6] B. G. Wachsmuth, *Interactive Real Analysis*, 2007. [Online]. Available: http://www.mathcs.org/analysis/reals/integ/proofs/rsums.html

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