What's all this about entanglement?

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Einstein-Podolsky-Rosen

Quantum mechanics: measurements on one particle dictate the state of the other particle.
\[ |\Psi\rangle = \frac{1}{\sqrt{2}} \left( |H\rangle_A |V\rangle_B - |V\rangle_A |H\rangle_B \right) \]

Reality #1

Reality #2
\[ |\Psi\rangle = \frac{1}{2} |H\rangle_A |V\rangle_B - \frac{1}{2} |H\rangle_A |H\rangle_B + \frac{1}{2} |V\rangle_A |V\rangle_B - \frac{1}{2} |V\rangle_A |H\rangle_B \]

... can be factored (separated)

\[ |\Psi\rangle = \frac{1}{2} (|H\rangle_A + |V\rangle_A) (|H\rangle_B - |V\rangle_B) \]

Separability \(\Rightarrow\) Not entangled
Spontaneous parametric down-conversion

One incoming photon $\rightarrow$ Two outgoing photons

Energy conservation:

$\omega_{\text{pump}} = \omega_{\text{signal}} + \omega_{\text{idler}}$

Momentum conservation:

$k_{\text{pump}} = k_{\text{signal}} + k_{\text{idler}}$

Degenerate phase matching conditions:

$\omega_{\text{signal}} = \omega_{\text{idler}} = \frac{1}{2} \omega_{\text{pump}}$ \hspace{1cm} \text{or} \hspace{1cm} \lambda_{\text{signal}} = \lambda_{\text{idler}} = 2\lambda_{\text{pump}}$

At NLC: $\lambda_{\text{pump}} = 355$ nm and $\lambda_{\text{signal}} = \lambda_{\text{idler}} = 710$ nm
Entanglement in momentum

Due to momentum conservation (remember \( p = \hbar k \)):

(summation \( \leftrightarrow \) integration)

\[
|\Psi\rangle_{\text{SPDC}} = \sum_n \alpha_n |k_n\rangle_A |k_p - k_n\rangle_B
\]

\[
= \alpha_1 |k_1\rangle_A |k_p - k_1\rangle_B + \alpha_2 |k_2\rangle_A |k_p - k_2\rangle_B
+ \alpha_3 |k_3\rangle_A |k_p - k_3\rangle_B + ... 
\]

where \( |k_p\rangle \) is the pump state and \( |k_n\rangle \) are plane wave states

Each term represent a different ‘reality’

The complete state does not factorize \( \Rightarrow \) the state is entangled
Entanglement in spatial modes

Entanglement in momentum basis ⇒ entanglement in any modal basis

Why? — 2 reasons:

- Different modal bases are all related by local unitary transformation
  \[ |M_m\rangle = U_{m,n} |k_n\rangle \text{ for } U_{m,n}U_{n,p}^\dagger = I_{m,p}. \]

- Local unitary transformation does not affect entanglement

  Example: \(|x\rangle = C |a\rangle - S |b\rangle; \ |y\rangle = S |a\rangle + C |b\rangle\), where \(C^2 + S^2 = 1\).

\[
\sqrt{2} |\psi\rangle = |x\rangle_A |x\rangle_B + |y\rangle_A |y\rangle_B \quad \leftarrow \text{entangled}
\]
\[
= (C |a\rangle_A - S |b\rangle_A) |x\rangle_B + (S |a\rangle_A + C |b\rangle_A) |y\rangle_B
\]
\[
= |a\rangle_A (C |x\rangle_B + S |y\rangle_B) + |b\rangle_A (-S |x\rangle_B + C |y\rangle_B)
\]
\[
= |a\rangle_A |a\rangle_B + |b\rangle_A |b\rangle_B \quad \leftarrow \text{entangled}
\]
Modes of an orbital angular momentum (OAM) basis:

$$M_{\text{OAM}}(r, \phi) = R_\ell(r) \exp(i\ell\phi)$$

in cylindrical coordinates

$\ell$ — azimuthal index (integer)

OAM is proportional to $\ell$

$R_\ell(r)$ — mode profile function

(examples: Laguerre-Gauss or Bessel-Gauss)
In terms of OAM modes:  
\[ |\Psi\rangle_{\text{SPDC}} = \sum_{\ell} \alpha_{\ell} |\ell\rangle_A |-\ell\rangle_B \]

Why? In thin-crystal limit (crystal length \(\ll\) pump Rayleigh range):
Three-way overlap:
\[ \alpha_{\ell} = \int M_p(x) M_s^*(x) M_i^*(x) \, d^2x \]

Assume pump is without OAM
\[ \Rightarrow \text{the azimuthal integration:} \]
\[ \int_0^{2\pi} \exp(-i\ell_s \phi) \exp(-i\ell_i \phi) \, d\phi = \begin{cases} 2\pi & \text{for } \ell_s = -\ell_i \\ 0 & \text{otherwise} \end{cases} \]

\[ \Rightarrow \text{the azimuthal indices } \ell_s \text{ and } \ell_i \text{ are anti-correlated} \]
Phase matching conditions

Depends on:

- Down-converted wavelength (usually degenerate)
- Dispersion properties (refractive index depends on wavelength)
- Birefringent medium (ordinary/extra-ordinary index)
- Polarization

Two types:

- **Type I phase matching**  
  \[ \Rightarrow \]  
  down-converted photons have the same polarization (both are ordinary)

- **Type II phase matching**  
  \[ \Rightarrow \]  
  down-converted photons have perpendicular polarizations (ordinary + extra-ordinary)
NLC experimental setup

Conditions: Type I, collinear and degenerate

BBO — Nonlinear crystal
M — Mirror
SLM — Spatial light modulator (reflective!)
SMF — Single mode fibre
APD — Avalanche photo diode
CC — Coincidence counter

Wavelength filters — not shown
4f - imaging (2 lenses each) — not shown
A spatial light modulator (SLM) is a kind of ‘mirror’ that introduces an arbitrary programmable phase function in the reflection of an incident optical beam

\[ f(x) \rightarrow f(x) \exp[i\theta_{\text{SLM}}(x)] \]

- \( f(x) \) — complex amplitude of input beam
- \( \theta_{\text{SLM}} \) — programmable phase function on the SLM

Example: helical phase with \( \ell = 3 \).

SLMs are the main controlling devices in our experiments.

Their versatility makes them very powerful.
Add a phase grating to the helical phase ($\text{OAM} = \ell$) on the SLM
Produce different diffraction orders: $n = \ldots, -2, -1, 0, 1, 2, \ldots$

Grating equation:

$$\sin(\theta_{\text{out}}) = \sin(\theta_{\text{in}}) + \frac{n\lambda}{d}$$

Each order adds (or subtracts) $\text{OAM} = n\ell$
Mode of single mode fibre (SMF) is approximately Gaussian. Coupling efficiency is computed by overlap integral:

\[ \eta_{SMF} = \int M_{in}(x) M_{SMF}^*(x) \, d^2x \]

Azimuthal integration:

\[ \int_0^{2\pi} \exp(i\ell_{in}\phi) \, d\phi = \begin{cases} 
2\pi & \text{for } \ell_{in} = 0 \\
0 & \text{otherwise} 
\end{cases} \]

⇒ only if diffraction order has \( \ell_{in} = 0 \)
will the light couple into the SMF.

Therefore, SLM + SMF gives one the ability to measure the OAM of an input beam.
How to measure the OAM spectrum and demonstrate entanglement:

Diagram:
- Laser
- BBO
- 50/50 beam splitter
- Signal
- SLM
- Idler
- M
- SMF
- APD
- CC

Graph:
- Coincidence counts
- \( \ell_A \) vs. \( \ell_B \)
- Color scale from 0 to 350
Quantum state

Pure single photon state:  \[ |\psi\rangle = \alpha |a\rangle + \beta |b\rangle \]
where \[ |\alpha|^2 + |\beta|^2 = 1 \] and \[ \langle a | b \rangle = 0 \].

All single photon qubit states lie on the Bloch sphere.

For bi-photon, multi-partite or higher-dimensional states (qudits) the state space is more complex.

Density operator:  \[ \rho = |\psi\rangle \langle \psi | \]

Mixed state:  \[ \rho = \sum_n |\psi_n\rangle P_n \langle \psi_n | \]
\[ P_n \] — probabilities
To quantify the entanglement of a state, one needs to know the exact quantum state.

Quantum state tomography

Tomography is a process whereby one reconstructs an object from different observations of the object.

Quantum state tomography: reconstructs a quantum state $\rho$ from different observations $\text{tr}\{P_n \rho\}$ where $P_n$ are different projections.
Full quantum state tomography for qubit (or qudit) states in terms of OAM:
Free-space quantum communication

The entangled photon pair is sent through the atmosphere and measured at the receiver.

Turbulence distorts the OAM modes
⇒ loss of entanglement
Single or multiple phase screen(s)

Single phase screen approach:

(a)  |  L  | (b)  
---   |     |     
Turbulent atmosphere  |     | Free space

Multiple phase screen approach:

ρ(z)  |  dz  | ρ(z+dz) 
---   |      |     
Turbulent atmosphere
NLC turbulence experiment

Measure decay of OAM entanglement with simulated turbulence using single phase screen approach:

![Diagram of the experiment](image)

**Entanglement decay**

![Graph showing concurrence vs. \(|\ell| = 1\) for different models](image)

- **Laser**
- **BBO**
- **50/50 beam splitter**
- **M**
- **SLM**
- **SMF**
- **APD**
- **CC**

**Random phase screen**
Ekert 91 protocol

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Prepare entangled state

Measure

Bob

Measure

Alice

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Mutually unbiased bases

QKD in higher dimensions needs mutually unbiased bases in higher dimensions

\[ |\langle \phi_{a,n} | \phi_{b,m} \rangle|^2 = \frac{1}{d} \quad \text{for} \quad a \neq b \]
Perform quantum key distribution by measuring the coincidence counts for different mutually unbiased bases.
At the NLC, we ...

- ... prepare OAM entangled photons, using SPDC
- ... perform projective measurements, using SLM + SMF
- ... measure OAM spectrum of SPDC output
- ... determine the quantum state, using full state tomography
- ... investigate scintillation of OAM entangled photons
- ... etc.

Stay tuned!