

# Quantitatively measuring the orbital angular momentum density of light

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## ABSTRACT

Although many techniques are efficient at measuring optical orbital angular momentum (OAM), they do not allow one to obtain a quantitative measurement for the OAM density across an optical field and instead only measure its global OAM. Numerous publications have demonstrated the transfer of local OAM to trapped particles by illustrating that particles trapped at different radial positions in an optical field rotate at different rotation rates. Measuring these rotation rates to quantitatively extract the OAM density is not only an indirect measurement but also a complicated experiment to execute. In this work we theoretically calculate and experimentally measure the OAM density of light, for both symmetric and non-symmetric optical fields. We outline a simple approach using only a spatial light modulator and a Fourier transforming lens to measure the OAM spectrum of an optical field and we test the approach on superimposed non-diffracting higher-order Bessel beams. We obtain quantitative measurements for the OAM density as a function of the radial position in the optical field for both symmetric and non-symmetric superpositions, illustrating good agreement with the theoretical prediction. The ability to measure the OAM distribution of optical fields has relevance in optical tweezing, and quantum information and processing.

**Keywords:** orbital angular momentum density, superimposed Bessel beams

## 1. INTRODUCTION

Laguerre-Gaussian beams [1], Bessel-Gaussian beams [2] and Airy beams [3] are all fields which carry orbital angular momentum (OAM) of  $l\hbar$  per photon as they have an azimuthal angular dependence of  $\exp(il\varphi)$  [1, 4], where  $l$  is the unbounded azimuthal mode index and  $\varphi$  is the azimuthal angle. Since the discovery of light beams carrying OAM, many methods have been developed for the detection of OAM modes, from the ‘fork’ hologram which projects the mode of interest into a detectable Gaussian mode [5] to the diffraction of apertures [6–8], where the 2D interference pattern is indicative of the OAM spectrum. Other techniques involve implementing the rotational frequency shift [9–11] which is a rather complex measurement, while recent work on dove prism interferometers [12] has resulted in the robust sorting of odd and even OAM modes [13]. An efficient mode sorter [14] has also been developed for the measurement of OAM states which has successfully been applied to Laguerre-Gaussian [15] and Bessel-Gaussian beams [16].

Even though the aforementioned techniques are efficient at measuring OAM modes, they do not allow one to obtain a quantitative measurement for the OAM density. To the best of our knowledge, the only attempts to make quantitative measurements of the OAM density have been made by measuring the rotation rates of particles trapped in optical tweezing systems [17, 18]. Not only is this a difficult experiment to conduct, but it is an indirect measurement. Another method uses the Doppler shifts of a rotating detector [19, 20] producing the reconstruction of the optical OAM spectrum.

In this manuscript we theoretically calculate and experimentally measure the OAM density of both symmetric and non-symmetric superpositions of non-diffracting Bessel beams [21–23]. We demonstrate a simple approach that requires only a spatial light modulator (SLM) and a lens to perform a radial azimuthal decomposition [22] or a modal decomposition into an appropriate basis [23] allowing for a quantitative reconstruction of the OAM density.

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## 2. THEORY

The amplitude of a non-symmetric superposition of two Bessel beams can be described as

$$u(r, \theta, z) = A_0 (J_l(q_1 r) \exp(i\Delta kz) \exp(il\theta) + \alpha_0 J_m(q_2 r) \exp(-i\Delta kz) \exp(im\theta)), \quad (1)$$

where  $J_l$  and  $J_m$  are the Bessel functions of orders  $l$  and  $m$ , respectively.  $q_1$  and  $q_2$  are the radial wave numbers,  $\Delta k$  is the difference between the two longitudinal wave numbers,  $\alpha_0$  is the ratio between the two component Bessel fields, and  $A_0$  is a normalization constant. The Poynting vector

$$\vec{S} = \varepsilon_0 c^2 \langle \vec{E}_{real} \times \vec{B}_{real} \rangle = \frac{i\omega\varepsilon_0}{4} (u \nabla u^* - u^* \nabla u) + \varepsilon_0 \omega k |u|^2 \hat{z}, \quad (2)$$

together with the total OAM density (in the direction of propagation,  $z$ )

$$L_z = \frac{1}{c^2} (r \times S)_z, \quad (3)$$

results in the total OAM density for the field given in Eq. (1) to be

$$L_z(r, \phi, z) = \frac{\varepsilon_0 \omega A_0^2}{2} \left( |J_l^2(q_1 r) + m \alpha_0^2 J_m^2(q_2 r) + (l+m) \alpha_0 \cos((l-m)\phi + 2\Delta kz) J_l(q_1 r) J_m(q_2 r) \right) \quad (4)$$

The term  $\Delta kz$  can be ignored as its effect on the optical field is only a constant phase-shift. From the theoretical OAM density a quantifiable measurement is determined as [22]

$$L_z(r, \phi, z) = \frac{\omega}{2c\eta} \left( \frac{\lambda f}{S_{Ring}} \right)^2 \left( |I_l(0) + m I_m(0) + (l+m) \cos((l-m)\phi + 2\Delta kz) \sqrt{I_l(0)} \sqrt{I_m(0)} \right) \quad (5)$$

Integrating over  $\phi$ , results in the average OAM density

$$\overline{L_z(r)} = \frac{\omega}{2c\eta} \left( \frac{\lambda f}{S_{Ring}} \right)^2 (|I_l(0) + m I_m(0)|). \quad (6)$$

Replacing the azimuthal order  $m$  with  $-l$ , the amplitude for a symmetric superposition of two Bessel beams, of opposite azimuthal order, is given by

$$u(r, \theta, z) = A_0 (J_l(q_1 r) \exp(i\Delta kz) \exp(il\theta) + \alpha_0 J_{-l}(q_2 r) \exp(-i\Delta kz) \exp(-il\theta)), \quad (7)$$

and the average OAM density as

$$\overline{L_z(r)} = \frac{l\omega}{2c\eta} \left( \frac{\lambda f}{S_{Ring}} \right)^2 (I_l(0) - I_{-l}(0)). \quad (8)$$

The symmetric and non-symmetric superpositions can be extended to consist of many Bessel beams and the experimental OAM densities can be extracted by extending the simple forms in Eqs (6) and (8).

Another technique to extract the OAM density of an optical field involves performing a modal decomposition [23]. Any optical field can be expressed in terms of modes, forming an orthogonal basis function

$$U(r) = \sum_{l=1}^N c_l \Psi_l(r). \quad (9)$$

Here  $c_l = \rho_l e^{i\phi_l}$  is the complex expansion coefficient with amplitude  $\rho_l$  and intermodal phase  $\Delta\phi_l$  (with respect to a reference phase),  $\Psi_l(r)$  is the  $l^{\text{th}}$  mode, and  $N$  the number of modes. The intensity and phase can be easily inferred from the modal amplitudes and phases.

The modal decomposition can be performed optically using correlation filters, which perform a correlation of the incident field with the modes that are encoded into the filter. An inner product measurement allows one to measure the power and relative phase of each individual mode. Measuring the power of a specific mode requires the complex conjugate of that mode to be programmed as the transmission function [24]:

$$T_l(r) = \Psi_l^*(r). \quad (10)$$

In the inner product measurement, this transmission function yields an intensity on the optical axis in the Fourier plane of the correlation filter that is proportional to  $\rho_l^2$ . Since the complete information about the optical field can be inferred from the modal decomposition, the Poynting vector can be calculated by Eq. 2 and subsequently the OAM density.

### 3. EXPERIMENTAL METHODOLOGY

The experimental setup for measuring the OAM density consisted of a section for the generation of the beam of interest (A) and their subsequent modal decomposition (B), as depicted in Fig. 1. A HeNe laser ( $\lambda \sim 633$  nm) was expanded through a 6 $\times$  telescope and directed onto the liquid crystal display of SLM<sub>1</sub>. The first SLM was encoded to produce various superposition fields using the concept of Durnin's ring-slit [25], but implemented digitally [26]. The resulting images of the Bessel fields were captured on a CCD camera or magnified with a 10 $\times$  objective and directed to SLM<sub>2</sub>. The incoming field was divided into 10 annular rings (Fig. 1 (f)) and the phase within each annular ring was varied as the complex conjugate of the azimuthal modes present in the incoming field. By measuring the on-axis intensity of the inner-product, the weighting coefficients,  $(\rho_l(r))^2$ , can be experimentally measured as a function of the radial co-ordinate and the azimuthal mode resulting in the OAM density can be quantitatively measured.

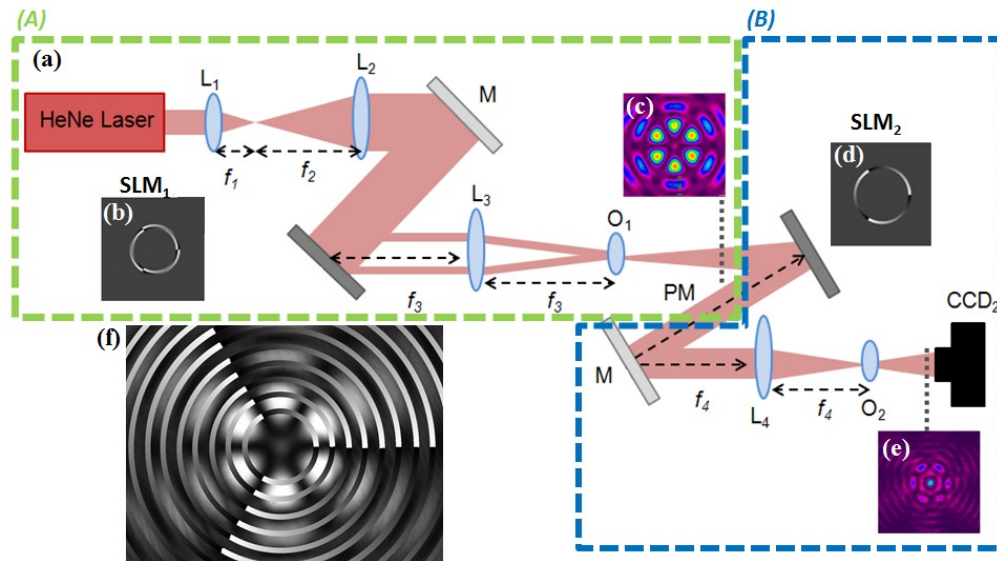


Fig. 1. A schematic of the experimental setup for measuring the OAM density as a function of the radial position. L: Lens ( $f_1 = 25$  mm;  $f_2 = 150$  mm;  $f_3 = 200$  mm and  $f_4 = 200$  mm); M: Mirror; SLM: Spatial light modulator; O: Objective; PM: Pop-up mirror; CCD: CCD camera. The objective, O2, was placed at the focus (or Fourier plane) of lens,  $L_4$ . The corresponding optical fields or holograms are represented at the appropriate planes. (f) An example of a field of interest subdivided into 10 annular rings each of azimuthal index  $l = 3$ .

The experimental setup for measuring the OAM density of light by performing a modal decomposition is depicted in Fig. 2 (a). To generate the various optical modes, for which we intend to calculate the OAM density, a HeNe laser was expanded through a  $8.3\times$  telescope and directed onto SLM<sub>1</sub>. By displaying their respective mode patterns on SLM<sub>1</sub> with the method described in [27, 28], having an intrinsic beam diameter of  $d = 1.5$  mm and a sinusoidal grating spacing of  $\Delta = 30$  pixels (an example is given in Fig. 2 (c)), the corresponding optical modes were generated (Fig. 2 (d)). The modes were relay imaged through a beam splitter (BS) to a near-field CCD camera, CCD<sub>1</sub> and a second SLM (SLM<sub>2</sub>) where the modal decomposition was performed. The modal weighting coefficients were found by executing an inner product of the mode of interest with a suitable match-filter (an example is given in Fig. 2 (e)) which was encoded with the same mode size as that on SLM<sub>1</sub>. The modal phases were extracted by the interference with a suitable reference mode [24]. The diffracted field from SLM<sub>2</sub> was Fourier transformed by lens  $L_5$  and the signal detected on CCD<sub>2</sub>, containing the on-axis intensity of the correlation channel, was used to infer the modal weighting coefficients.

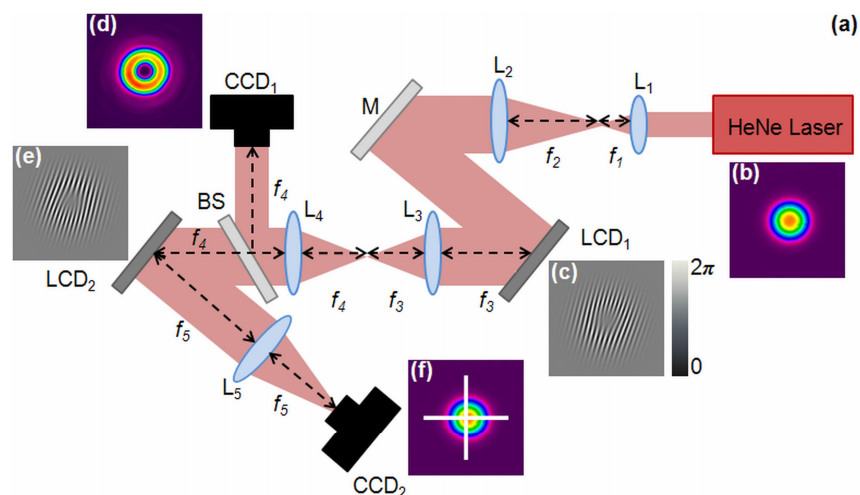


Fig. 2. (a) A schematic of the experimental setup for measuring the OAM density of light by performing a modal decomposition. L: Lens ( $f_1 = 15$  mm;  $f_2 = 125$  mm;  $f_3 = 500$  mm;  $f_4 = 500$  mm and  $f_5 = 300$  mm); M: Mirror; LCD:

Liquid Crystal Display; BS: Beam Splitter; CCD: CCD Camera. (b) The Gaussian beam used to illuminate SLM<sub>1</sub>. (c) The digital hologram used to generate the optical mode of interest (d). The digital hologram (e) used to extract the weighting of the mode by detecting the on-axis intensity at the plane of the inner product (f).

#### 4. RESULTS AND DISCUSSION

SLM<sub>1</sub> was first encoded with a ring-slit separated into two ring-slits (of equal widths), each possessing an azimuthal phase of equal order but opposite handedness, ( $l_{\text{inner}} = 3$  and  $l_{\text{outer}} = -3$ ) and then later with unequal azimuthal phases ( $l_{\text{inner}} = 3$  and  $l_{\text{outer}} = 4$ ). In the case that SLM<sub>1</sub> was encoded with two ring-slits, where the orders of the two azimuthal phases are of equal but opposite handedness, a ‘petal’-structure was produced, where the number of ‘petals’ is denoted by  $2|l|$ . The match-filter (encoded on SLM<sub>2</sub>) was programmed digitally so that the radius of the ring-slit, as well as the azimuthal phase within the ring-slit can be dynamically addressed. This allows one to radially locate where in the optical field one wishes to measure the OAM density. The OAM density for a particular radial position can then be measured directly from Eqs (6) and (8) by measuring the on-axis intensity of the inner-product. The measured OAM densities as a function of the radial position for a symmetric and non-symmetric optical field are given in Fig. 3 (a) and (b), respectively.

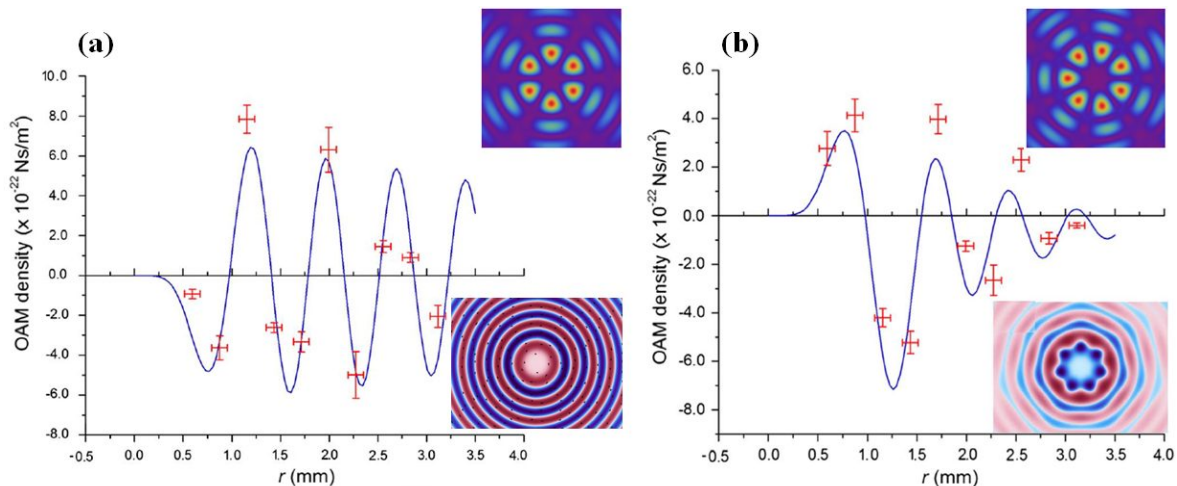


Fig. 3. (a) The theoretical (blue curve) and experimentally measured (red points) OAM density for a symmetric superposition. (b) The theoretical (blue curve) and experimentally measured (red points) OAM density for a non-symmetric superposition. Inserts denote the theoretically calculated field (top) and a density plot of the OAM density (bottom) where red denotes negative OAM and blue denotes positive.

In implementing the experimental setup described in Fig. 2 which involved performing a modal decomposition using the same mode size as that encoded on SLM<sub>1</sub>, we examined a superposition of two Laguerre-Gaussian beams  $LG_{0,1} + LG_{0,-2} \exp(-i\pi/3)$  with an intermodal phase shift of  $\Delta\phi = -\pi/3$ . The results are depicted in Fig. 4. The reconstructed intensity (Fig. 4 (a)) reveals the 3-lobe structure formed by the interference of the two modes. The Poynting vector spirals around four rotation centers depicting the phase singularities (marked with red dots in Fig. 4 (a)). Figure 4 (b) and (c) depict the measured and calculated OAM density, which have a correlation of 92 %. Since the azimuthal indices of the field of interest are both positive and negative ( $l = 1$  and  $l = -2$ ), the OAM density possesses both positive and negative values. The structure of the OAM density resembles the triangular symmetry of the intensity distribution with maxima in the vicinity of the phase singularities. A similar measurement can easily be performed on superpositions of Bessel beams.

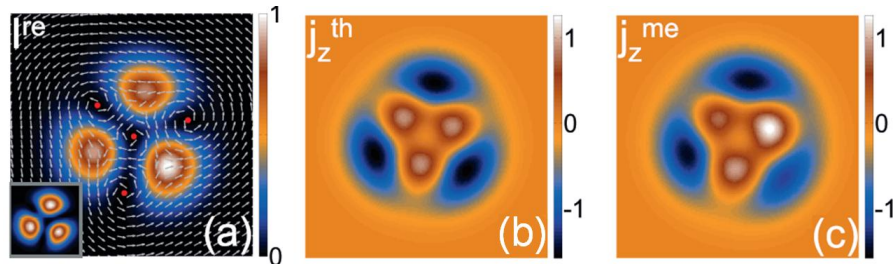


Fig. 4. (a) The reconstructed intensity and transverse Poynting vectors (red dots mark position of the phase singularities). (b) The theoretical OAM density and (c) the measured OAM density. Inset in (a) depicts measured beam intensity.

## 5. CONCLUSION

We have used the Poynting vector approach to derive expressions for the OAM density for optical fields. For both a symmetric and a non-symmetric superposition of non-diffracting Bessel beams, we obtain quantitative measurements for the OAM density as a function of the radial position and illustrate good agreement with the theoretical prediction. Although the global OAM is zero, the local OAM spectrum changes radially across the beam. Another technique that was implemented is based on correlation filters to perform a modal decomposition with subsequent reconstruction of the optical field, allowing one to infer the Poynting vector and the OAM density. Both methods require only an appropriate hologram, a lens and a single point detector.

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