Stochastic singular optics

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our future through science

Contents

Oxymoron?

- ▷ Notion: partial coherence destroys vortices
- ▷ Hence, no vortices in stochastic optical fields
- ▷ However, not interested in individual vortices
- ▷ Instead, study related quantities:
  - — vortex distributions
  - — topological charge distributions
  - — phase gradient
  - — orbital angular momentum
- ▷ Not considering polarization

Definition of vortex distributions

- 2D (transverse) plane as a slice (foliation) through 3D optical field, perpendicular to propagation (or arbitrary observation) direction
- Vortex is first order zero of a 2D complex field (in 3D vortex becomes a line)
- Vortex density: number of vortices per unit area on the 2D plane, function of transverse coordinates (x, y)
- Also function of z: propagation distance z replaces time as an independent variable of causal progression
- Two types of vortices: topological charge ± 1 (higher order are unstable). Positive and negative vortex densities $n_{\rm p}(x, y, z)$ and $n_{\rm n}(x, y, z)$
- Vortex density: $V = n_{\rm p} + n_{\rm n}$
- Topological charge density: $T = n_{\rm p} - n_{\rm n}$

Subfields of SSO

- ▷ Homogeneous, normally distributed (speckle):
  $\Rightarrow$ stationary (in equilibrium)
  — statistics of vortex distributions
  — topology of vortex lines
  — coherence vortices
- ▷ Homogeneous, not normally distributed:
  $\Rightarrow$ transient evolution in V, while $T = 0$
  — vortices in scintillated optical field in random media
  — removal of continuous phase $\rightarrow$ dip in V
- ▷ Inhomogeneous, normally distributed:
  $\Rightarrow$ transient evolution in V and T
  — diffusion, phase and amplitude drift of V and T
- ▷ Inhomogeneous, not normally distributed:
  — perhaps one day

Numerical simulations

- ▷ Produce input field
  - — complex sampled array
  - — random, but with correlations
- ▷ Perform beam propagation

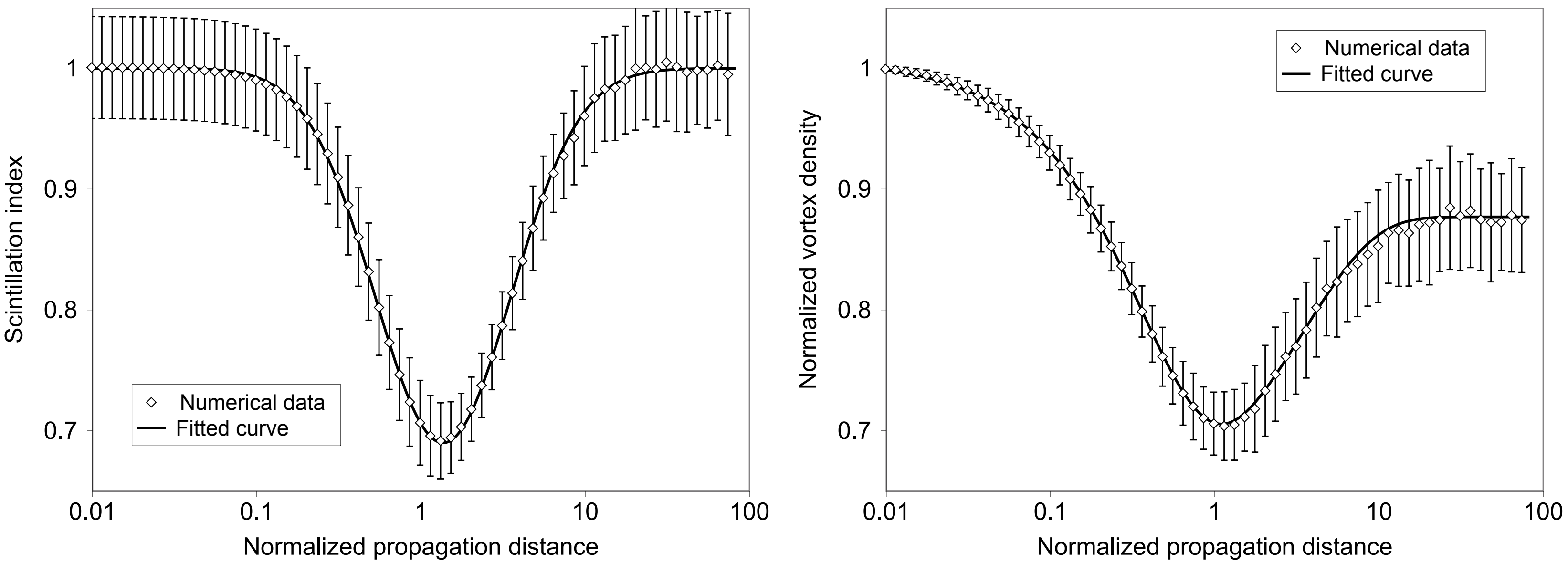
$$g(x, y, z = 0) \rightarrow G = \mathcal{F}\{g\} \rightarrow g(x, y, z) = \mathcal{F}^{-1}\{G\psi(z)\}$$

- ▷ Extract information (locate vortices)
- ▷ Perform averaging

Can be used for fields that are normally distributed or not normally distributed

New physics

Homogeneous, not normally distributed[a]

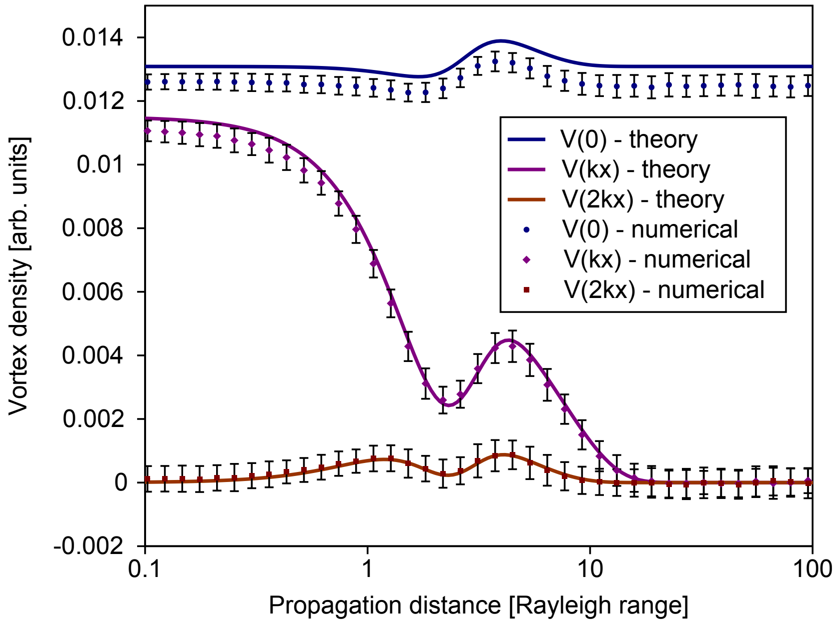


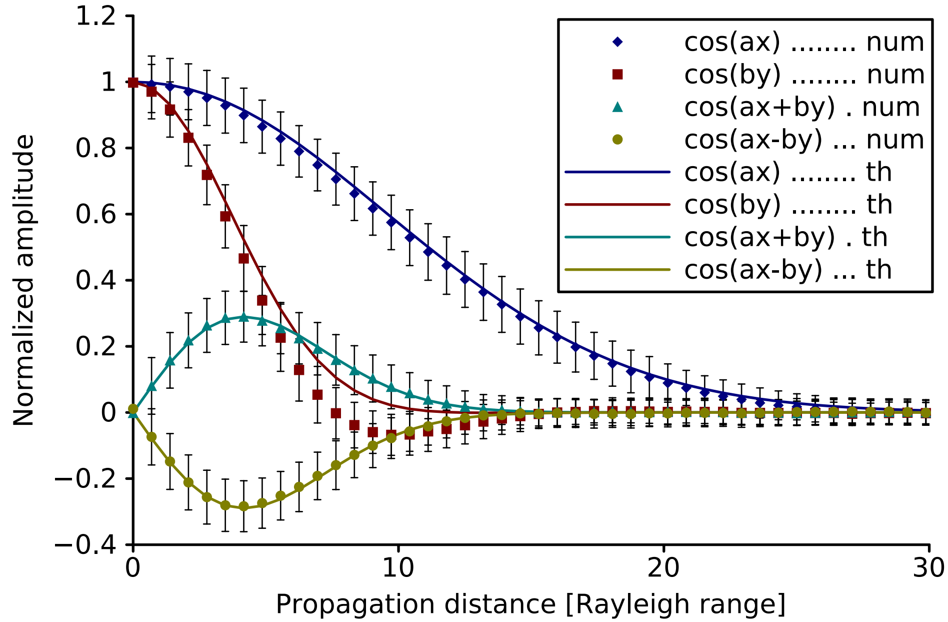
[a] M. Chen and F.S. Roux, Phys. Rev. A, **80**, 013824 (2009);

— J. Opt. Soc. Am. A, **27**, 2138–2143 (2010).

New physics (cont.)

Inhomogeneous, normally distributed[a]





[a] F.S. Roux, J. Opt. Soc. Am. A, **28**, 621–626 (2011);

— Opt. Comm., **285**, 947–952 (2012).

Statistical optics calculations

Assume normal distribution

Two step process:

- ▷ Expression of local quantity
  in terms of field correlation functions
  - — done once
  - — expressions in their most general form
- ▷ Calculation of field correlation functions
  - — for every stochastic optical field
  - — examples to be studied

General correlation functions

Use non-local correlation function to compute local correlation functions

- Mutual coherence function[a]

$$\Gamma_{\text{full}}(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2) = \langle g(\mathbf{x}_1, t_1)\overline{g}(\mathbf{x}_2, t_2)\rangle$$

- Monochromatic (drop time) and paraxial (set z equal)

$$\Gamma(x_1, y_1, x_2, y_2, z) = \langle g(x_1, y_1, z)\overline{g}(x_2, y_2, z)\rangle$$

- Intensity: $I(x, y, z) = \langle g\overline{g}\rangle = \Gamma(x, y, x, y, z)$
- General two-point field correlation functions

$$\langle g\overline{g}_x\rangle = [\partial_x\langle g(u, v, z)\overline{g}(x, y, z)\rangle]_{u=x,v=y} = [\partial_x\Gamma(u, v, x, y, z)]_{u=x,v=y}$$

[a] J. W. Goodman, *Statistical optics* (Wiley-Interscience, New York, 1985).

Expectation values

- Consider optical field and its first derivatives
- Expectation value of W as a function of random variables $\mathbf{q} = \{q_n\}$

$$\langle W \rangle = \int W(\mathbf{q}) P_{\mathbf{q}}(\mathbf{q})\, \mathrm{d}^6 q$$

- Joint probability density function:

$$P_{\mathbf{q}}(\mathbf{q}) = \frac{\exp\left(-\mathbf{Q}^{\dagger} M_1^{-1} \mathbf{Q}\right)}{\pi^3 \det(M_1)}$$

where $\mathbf{Q} = [q_1 + \mathrm{i}q_2, q_3 + \mathrm{i}q_4, q_5 + \mathrm{i}q_6]^T$

- Covariance matrix:

$$M_1 = \begin{bmatrix} \langle g\overline{g} \rangle & \langle g_x\overline{g} \rangle & \langle g_y\overline{g} \rangle \\ \langle g\overline{g}_x \rangle & \langle g_x\overline{g}_x \rangle & \langle g_y\overline{g}_x \rangle \\ \langle g\overline{g}_y \rangle & \langle g_x\overline{g}_y \rangle & \langle g_y\overline{g}_y \rangle \end{bmatrix}$$

Vortex density

- Vortex density as an average[a]

$$V(\mathbf{x}) = \int |q_3 q_6 - q_4 q_5| \left[P_{\mathbf{q}}(\mathbf{q})\right]_{q_1=q_2=0} \ \mathrm{d}^4 q$$

- After evaluating q-integrals:

$$V(\mathbf{x}) = \frac{2\langle g\overline{g}\rangle \det(M_1) - B^2}{2\pi \langle g\overline{g}\rangle^2 \sqrt{4\langle g\overline{g}\rangle \det(M_1) - B^2}}$$

where $\det(M_1) = \langle g\overline{g}\rangle(\langle g_x\overline{g}_x\rangle\langle g_y\overline{g}_y\rangle - \langle g_y\overline{g}_x\rangle\langle g_x\overline{g}_y\rangle)$
$-\langle g_x\overline{g}_x\rangle\langle g_y\overline{g}\rangle\langle g\overline{g}_y\rangle - \langle g_y\overline{g}_y\rangle\langle g_x\overline{g}\rangle\langle g\overline{g}_x\rangle + \langle g_x\overline{g}_y\rangle\langle g_y\overline{g}\rangle\langle g\overline{g}_x\rangle$
$+\langle g_y\overline{g}_x\rangle\langle g_x\overline{g}\rangle\langle g\overline{g}_y\rangle$ and $B = \langle g\overline{g}\rangle(\langle g_x\overline{g}_y\rangle - \langle g_y\overline{g}_x\rangle)$
$+\langle g\overline{g}_x\rangle\langle g_y\overline{g}\rangle - \langle g_x\overline{g}\rangle\langle g\overline{g}_y\rangle$

[a] M.V. Berry, J. Phys. A: Math. Gen. **11**, 27–37 (1978);

M.V. Berry and M.R. Dennis, Proc. R. Soc. Lond. A **456**, 2059–2079 (2000);

F.S. Roux, J. Opt. Soc. Am. A **28**, 621–626 (2011).

Topological charge density

- Topological charge density as an average

$$T(\mathbf{x}) = \int (q_3 q_6 - q_4 q_5) \left[P_{\mathbf{q}}(\mathbf{q})\right]_{q_1=q_2=0} \, \mathrm{d}^4 q$$

- After evaluating q-integrals:

$$T(\mathbf{x}) = \frac{\mathrm{i}B}{2\pi\langle g\overline{g}\rangle^2}$$

where

$$B = \langle g\overline{g}\rangle(\langle g_x\overline{g}_y\rangle - \langle g_y\overline{g}_x\rangle) + \langle g\overline{g}_x\rangle\langle g_y\overline{g}\rangle - \langle g_x\overline{g}\rangle\langle g\overline{g}_y\rangle$$

Local phase gradient

- Phase of an optical field $g(\mathbf{x}) = A(\mathbf{x})\exp[\mathrm{i}\theta(\mathbf{x})]$:

$$\theta(\mathbf{x}) = \frac{-\mathrm{i}}{2}\ln\left[\frac{g(\mathbf{x})}{\overline{g}(\mathbf{x})}\right]$$

- Phase gradient

$$\nabla\theta(\mathbf{x}) = -\mathrm{i}\frac{\overline{g}(\mathbf{x})\nabla g(\mathbf{x}) - g(\mathbf{x})\nabla\overline{g}(\mathbf{x})}{2|g(\mathbf{x})|^2}.$$

where $\nabla = \hat{x}\partial_x + \hat{y}\partial_y$

- Local phase gradient as an average

$$\mathbf{F}(\mathbf{x}) = \int \frac{(q_3 q_2 - q_4 q_1)\hat{x} + (q_5 q_2 - q_6 q_1)\hat{y}}{q_1^2 + q_2^2} P_{\mathbf{q}}(\mathbf{q})\ \mathrm{d}^6 q$$

- After evaluating q-integrals:

$$\mathbf{F}(\mathbf{x}) = \mathrm{i}\frac{(\langle g\overline{g}_x\rangle - \langle g_x\overline{g}\rangle)\hat{x} + (\langle g\overline{g}_y\rangle - \langle g_y\overline{g}\rangle)\hat{y}}{2\langle g\overline{g}\rangle} = \frac{\mathbf{v}_2}{2\langle g\overline{g}\rangle}$$

Magnitude of local phase gradient

- Magnitude of the local phase gradient as an average

$$F(\mathbf{x}) = \int \frac{\sqrt{(q_3 q_2 - q_4 q_1)^2 + (q_5 q_2 - q_6 q_1)^2}}{q_1^2 + q_2^2} P_{\mathbf{q}}(\mathbf{q}) \, \mathrm{d}^6 q$$

- After evaluating q-integrals:

$$F(\mathbf{x}) = \frac{\sqrt{G+H}}{8\pi\sqrt{2}\langle g\overline{g}\rangle} \mathrm{E}\left(\sqrt{\frac{2H}{G+H}}\right)$$

where E$(\cdot)$ is the complete elliptic integral of the second kind, and G and H are rather complicated expressions.

- Clearly, $F(\mathbf{x}) \neq |\mathbf{F}(\mathbf{x})|$, because $\langle |g| \rangle \neq |\langle g \rangle|$.

Magnitude of phase gradient (cont.)

$$G = 4\langle g\overline{g}\rangle(\langle g_x\overline{g}_x\rangle + \langle g_y\overline{g}_y\rangle) - (\langle g_x\overline{g}\rangle + \langle g\overline{g}_x\rangle)^2 - (\langle g_y\overline{g}\rangle + \langle g\overline{g}_y\rangle)^2$$

$$\begin{aligned} H = \Big(&16\langle g\overline{g}\rangle^2 \left[(\langle g_x\overline{g}_x\rangle - \langle g_y\overline{g}_y\rangle)^2 + (\langle g_x\overline{g}_y\rangle + \langle g_y\overline{g}_x\rangle)^2\right] \\ &-8\langle g\overline{g}\rangle \Big\{ \left[(\langle g_x\overline{g}\rangle + \langle g\overline{g}_x\rangle)^2 - (\langle g_y\overline{g}\rangle + \langle g\overline{g}_y\rangle)^2\right](\langle g_x\overline{g}_x\rangle - \langle g_y\overline{g}_y\rangle) \\ &+2(\langle g_x\overline{g}_y\rangle + \langle g_y\overline{g}_x\rangle)(\langle g\overline{g}_y\rangle + \langle g_y\overline{g}\rangle)(\langle g_x\overline{g}\rangle + \langle g\overline{g}_x\rangle)\Big\} \\ &+ \left[(\langle g_x\overline{g}\rangle + \langle g\overline{g}_x\rangle)^2 + (\langle g_y\overline{g}\rangle + \langle g\overline{g}_y\rangle)^2\right]^2\Big)^{1/2} \end{aligned}$$

Null crossing line density

- Line density of null crossings

$$N(\mathbf{x}) = \frac{1}{A}\int_A \delta(g_q)\sqrt{\left(\partial_x g_q\right)^2 + \left(\partial_y g_q\right)^2}\,\mathrm{d}x\mathrm{d}y$$

  where g_q represents either g_r or g_i.

- After evaluating 3 q-integrals:

$$N(\mathbf{x}) = \frac{\sqrt{G+H}}{2\pi\sqrt{2}\langle g\,\overline{g}\rangle}\mathrm{E}\left(\sqrt{\frac{2H}{G+H}}\right)$$

- $N(\mathbf{x}) = 4F(\mathbf{x})$, because average separation distance between null crossings is proportional to the magnitude of the local phase gradient

Quantities with higher derivatives

- For up the second derivatives covariance matrix expands to rank 6:

$$M_2 = \begin{bmatrix} \langle g\overline{g}\rangle & \langle g_x\overline{g}\rangle & \langle g_y\overline{g}\rangle & \langle g_{xx}\overline{g}\rangle & \langle g_{xy}\overline{g}\rangle & \langle g_{yy}\overline{g}\rangle \\ \langle g\overline{g}_x\rangle & \langle g_x\overline{g}_x\rangle & \langle g_y\overline{g}_x\rangle & \langle g_{xx}\overline{g}_x\rangle & \langle g_{xy}\overline{g}_x\rangle & \langle g_{yy}\overline{g}_x\rangle \\ \langle g\overline{g}_y\rangle & \langle g_x\overline{g}_y\rangle & \langle g_y\overline{g}_y\rangle & \langle g_{xx}\overline{g}_y\rangle & \langle g_{xy}\overline{g}_y\rangle & \langle g_{yy}\overline{g}_y\rangle \\ \langle g\overline{g}_{xx}\rangle & \langle g_x\overline{g}_{xx}\rangle & \langle g_y\overline{g}_{xx}\rangle & \langle g_{xx}\overline{g}_{xx}\rangle & \langle g_{xy}\overline{g}_{xx}\rangle & \langle g_{yy}\overline{g}_{xx}\rangle \\ \langle g\overline{g}_{xy}\rangle & \langle g_x\overline{g}_{xy}\rangle & \langle g_y\overline{g}_{xy}\rangle & \langle g_{xx}\overline{g}_{xy}\rangle & \langle g_{xy}\overline{g}_{xy}\rangle & \langle g_{yy}\overline{g}_{xy}\rangle \\ \langle g\overline{g}_{yy}\rangle & \langle g_x\overline{g}_{yy}\rangle & \langle g_y\overline{g}_{yy}\rangle & \langle g_{xx}\overline{g}_{yy}\rangle & \langle g_{xy}\overline{g}_{yy}\rangle & \langle g_{yy}\overline{g}_{yy}\rangle \end{bmatrix}$$

- Higher complexity: $\det(M_1) = 6$, but $\det(M_2) = 720$.
- Examples of quantities: distributions of the Poincarè-Hopf indices and probability density for annihilation and creation events

Complexity in SSO

The predominant challenge is the complexity

- $\triangleright$ Expression often contain large polynomials in correlation functions
- $\triangleright$ Higher derivatives enhance the complexity significantly
- $\triangleright$ Dynamics require higher derivatives
- $\triangleright$ $\rightarrow$ exploit coordinate invariance
- $\triangleright$ The choice of coordinate system is arbitrary (one can rotate the x- and y-axes by an arbitrary angle)
- $\triangleright$ Expression of the expectation value of quantities are not affected by coordinate transformation
- $\triangleright$ Coordinate rotation: SO(2) Lie group
- $\triangleright$ $\Rightarrow$ Expressions are (or consist of) SO(2) singlets

Coordinate transformations

- Reducible SO(2) transformation of covariant matrix:

$$M_1 \to O M_1 O^{-1},$$

where

$$O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

α — rotation angle

- Irreducible SO(2) transformation, example:

$$\begin{matrix} \langle g\overline{g}_x \rangle \\ \langle g\overline{g}_y \rangle \end{matrix} \to \begin{matrix} \langle g\overline{g}_x \rangle \cos(\alpha) - \langle g\overline{g}_y \rangle \sin(\alpha) \\ \langle g\overline{g}_x \rangle \sin(\alpha) + \langle g\overline{g}_y \rangle \cos(\alpha) \end{matrix}$$

- Defined SO(2) singlets i.t.o. correlation functions (τ_n's)

Expressions i.t.o. SO(2) singlets

- ▷ Topological charge density $$T = \frac{\tau_0\tau_2 - \tau_6}{2\pi\tau_0^2}$$

- ▷ Vortex density

$$V = \frac{2\tau_0(\tau_0\tau_7 - \epsilon_1) + (\tau_0\tau_2 - \tau_6)^2}{2\pi\tau_0^2\sqrt{4\tau_0(\tau_0\tau_7 - \epsilon_1) + (\tau_0\tau_2 - \tau_6)^2}}$$

- ▷ Relationship among V, T and Q

$$V = \frac{Q + 2T^2}{2\sqrt{Q+T^2}} = \frac{1}{2}\sqrt{Q+T^2} + \frac{T^2}{2\sqrt{Q+T^2}}$$

- ▷ Determinant

$$Q = \frac{\det(M_1)}{\pi^2\tau_0^3} = \frac{\tau_0\tau_7 - \epsilon_1}{\pi^2\tau_0^3}$$

Curl of phase gradient

Using the following identities

- $\nabla \times \mathbf{v}_2 = 2\tau_2 \hat{z} = \mathrm{i}2(\langle g_x \overline{g}_y \rangle - \langle g_y \overline{g}_x \rangle)\hat{z}$
- $\nabla \tau_0 \equiv \mathbf{v}_1 = (\langle g_x \overline{g} \rangle + \langle g \overline{g}_x \rangle)\hat{x} + (\langle g_y \overline{g} \rangle + \langle g \overline{g}_y \rangle)\hat{y}$
- $\mathbf{v}_1 \times \mathbf{v}_2 = 2\tau_6 \hat{z} = \mathrm{i}2(\langle g_x \overline{g} \rangle \langle g \overline{g}_y \rangle - \langle g_y \overline{g} \rangle \langle g \overline{g}_x \rangle)\hat{z}$

one can show that

$$\begin{aligned}
\nabla \times \mathbf{F} &= \nabla \times \left(\frac{\mathbf{v}_2}{2\tau_0} \right) = \frac{\nabla \times \mathbf{v}_2}{2\tau_0} - \frac{\nabla \tau_0 \times \mathbf{v}_2}{2\tau_0^2} \\
&= \frac{\tau_2 \hat{z}}{\tau_0} - \frac{\mathbf{v}_1 \times \mathbf{v}_2}{2\tau_0^2} = \frac{(\tau_0 \tau_2 - \tau_6)\hat{z}}{\tau_0^2} \\
&= 2\pi\, T \hat{z}
\end{aligned}$$

Intensity transport

- ▷ The z-derivative to the intensity $\tau_0 = \langle g\overline{g}\rangle$

$$\partial_z \tau_0 = \frac{-\mathrm{i}}{2k}\left(\langle g_{xx}\overline{g}\rangle - \langle g\overline{g}_{xx}\rangle + \langle g_{yy}\overline{g}\rangle - \langle g\overline{g}_{yy}\rangle\right) = \frac{\nabla\cdot\mathbf{v}_2}{2k}$$

- ▷ The divergence of the phase gradient

$$\nabla\cdot\mathbf{F} = \nabla\cdot\left(\frac{\mathbf{v}_2}{2\tau_0}\right) = \frac{\nabla\cdot\mathbf{v}_2}{2\tau_0} - \frac{\nabla\tau_0\cdot\mathbf{v}_2}{2\tau_0^2} = \frac{k\partial_z\tau_0}{\tau_0} - \frac{\nabla\tau_0\cdot\mathbf{F}}{\tau_0}$$

- ▷ The intensity transport equation[a]

$$k\partial_z\tau_0 = \tau_0\nabla\cdot\mathbf{F} + (\nabla\tau_0)\cdot\mathbf{F} = \nabla\cdot(\tau_0\mathbf{F})$$

[a] M.R. Teague, J. Opt. Soc. Am. **72**, 1199–1209 (1982).

Summary

- ▷ Introduced the field (and subfields) of Stochastic Singular Optics
- ▷ Numerical simulations reveals some new physics — things we don't understand yet
- ▷ Statistical optics calculations provide expressions for various quantities
- ▷ Complexity of these expressions and their derivatives is a challenges in Stochastic Singular Optics, but ...
- ▷ Coordinate invariance gives SO(2) singlets in terms of which expressions become simpler, as a result ...
- ▷ Identified some (algebraic and differential) relationships among the different quantities