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# RAYLEIGH-LOVE MODEL OF LONGITUDINAL VIBRATIONS OF CONICAL AND EXPONENTIAL RODS: EXACT SOLUTIONS AND NUMERICAL SIMULATION BY THE METHOD OF LINES

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New exact solutions of equations of longitudinal vibration of conical and exponential rod are obtained for the Rayleigh-Love model. These solutions are used as reference results for checking accuracy of the method of lines. It is shown that the method of lines generates solutions, which are very close to those that are predicted by the exact theory. It is also shown that the accuracy of the method of lines is improved with increasing the number of intervals on the rod. Reliability of numerical methods is very important for obtaining approximate solutions of physical and technical problems. In the present paper we consider the Rayleigh-Love model of longitudinal vibrations of rods with conical and exponential cross-sections. It is shown that exact solution of the problem of longitudinal vibration of the conical rod is obtained in Legendre spherical functions and the corresponding solution for the rod of exponential cross-section is expressed in the Gauss hypergeometric functions. General solution of these problems is expressed in terms of the Green function. For numerical solution of the problem we use the method of lines. By means of this method the partial differential equations describing the dynamics of the Rayleigh-Love rod are reduced to a system of ordinary differential equations. For checking of accuracy of the numerical solution we chose special initial conditions, namely we assume that initial longitudinal displacements of the rod are proportional to one of eigenfunction of the system and initial velocities are zero. In this case vibrations of every point of the rod are harmonic and their amplitudes are equal to the initial displacements. Periods of these vibrations, obtained by the method of lines are estimated and compared with the theoretically predicted eigenvalues of the rod, thus giving us estimations of accuracy of the numerical procedures.

## 1. Introduction

Reliability of numerical methods is very important for obtaining approximate solutions of physical and technical problems. That is why it is necessary to test these solutions whenever it is possible using exact solutions, obtained for some special cases. In the present paper we consider the Rayleigh-Love model [1] of longitudinal vibrations of rods with conical and exponential cross-sections. It is shown that exact solution of the problem of longitudinal vibration of the conical rod is obtained in Legendre spherical functions and the corresponding solution for the rod of exponential cross-section is expressed in the Gauss hypergeometric functions. For numerical solution of the problem we use the method of lines [2].

## 2. Exact solution of equations of the conical rod

Let us consider a rod of length $l$ and assume that its physical parameters such as mass density $(\rho)$, modulus of elasticity $(E)$ and Poisson ratio $(\eta)$ are constant, but radius of cross-section is variable and depends on longitudinal coordinate $(x)$ of the rod: $r=r(x)$. In this case area of cross-section of the rod $(S=S(x))$ and its polar moment of inertia $(I_p=I_p(x))$ are also variable. In the case of circular cross-section $S(x)=\pi r^2(x)$ and $I_p(x)=\pi r^4(x)/2$. Equation of longitudinal vibration [1] for longitudinal displacement $u(x,t)$ is as follows:

$$\rho S(x)\frac{\partial^2 u(x,t)}{\partial t^2}-\rho\eta^2\frac{\partial}{\partial x}\left[I_p(x)\frac{\partial^3 u(x,t)}{\partial t^2\partial x}\right]-E\frac{\partial}{\partial x}\left[S(x)\frac{\partial u(x,t)}{\partial x}\right]=F(x,t) \qquad (1)$$

Let us consider a steady-state vibration $u(x,t)=U(x)\cdot e^{i\omega t}$ ($i^2=-1$). In this case the corresponding to (1) homogeneous equation is:

$$\rho\omega^2\left\{S(x)U(x)-\eta^2\frac{d}{dx}\left[I_p(x)\frac{dU(x)}{dx}\right]\right\}+E\frac{d}{dx}\left[S(x)\frac{dU(x)}{dx}\right]=0 \qquad (2)$$

If the generatrix of conical surface of the rod is described by equation $r(x)=k\left(x-x_p\right)=k\overline{x}$, where $x_p$ is coordinate of the pole of the cone, $\overline{x}=x-x_p$, then $S(x)=\pi k^2\overline{x}^2$, $I_p(x)=\pi k^4\overline{x}^4/2$ and equation (2) is rewritten as follows:

$$\left(1-\mu^2\overline{x}^2\right)\frac{d^2U(\overline{x})}{d\overline{x}^2}+\frac{2\left(1-2\mu^2\overline{x}^2\right)}{\overline{x}}\frac{dU(\overline{x})}{d\overline{x}}+\left(\frac{\omega}{c}\right)^2U(\overline{x})=0 \qquad (3)$$

where $c=\sqrt{E/\rho}$ - speed of wave propagation in cylindrical rod in accordance with the classical theory, and $\mu=\dfrac{\eta k\omega}{c\sqrt{2}}$ is the wavenumber of the conical rod which has dimension $m^{-1}$.

Introducing new dimensionless variable $z=\mu\overline{x}$, considering new function $V(z)=U\left(\dfrac{z}{\mu}\right)=\dfrac{W(z)}{z}$ we transform (3) to equation:

$$\left(1-z^2\right)\frac{d^2W(z)}{dz^2}-2z\frac{dW(z)}{dz}+\sigma(\sigma+1)W(z)=0 \tag{4}$$

where $\sigma=-\dfrac{1}{2}+\sqrt{\dfrac{9}{4}+\dfrac{2}{(\eta k)^2}}$. Equation (4) is the Legendre equation which has solution

$$W(z)=\overline{C}_1P_\sigma(z)+\overline{C}_2Q_\sigma(z) \tag{5}$$

where $P_\sigma(z)$, $Q_\sigma(z)$ are Legendre functions of the first and second kind and $\overline{C}_{1,2}$ are arbitrary constants. In original variables solution of the problem of the Rayleigh-Love longitudinal vibration of the conical rod is rewritten as follows:

$$U(x)=\frac{\overline{C}_1}{\mu}\frac{P_\sigma\left[\mu\left(x-x_p\right)\right]}{x-x_p}+\frac{\overline{C}_2}{\mu}\frac{Q_\sigma\left[\mu\left(x-x_p\right)\right]}{x-x_p} \tag{6}$$

## 3. Exact solution of equations of the exponential rod

Let us now consider the Rayleigh-Love rod with the exponential generatrix so that radius of its cross-section is $r(x)=k\cdot e^{\alpha\cdot x}$. In this case area of cross-section is $S(x)=\pi k^2e^{2\alpha x}$ and polar moment of inertia $I_p(x)=\dfrac{\pi k^4e^{4\alpha x}}{2}$. In this case equation (2) is transformed to the following form:

$$\left(1-\chi e^{2\alpha x}\right)\frac{d^2U(x)}{dx^2}+2\alpha\left(1-2\chi e^{2\alpha x}\right)\frac{dU(x)}{dx}+\left(\frac{\omega}{c}\right)^2U(x)=0 \tag{7}$$

where $\chi=\dfrac{1}{2}\left(\dfrac{\eta k\omega}{c}\right)^2$. Exact solution of equation (7) could be obtained by means of it transformation to the Gauss hypergeometric equation in two steps. At the first step we make transformation $U(x)=V(x)e^{\beta x}$, where $\beta$ is constant, which will be specially selected further. After this transformation equation (7) is rewritten as

$$\begin{aligned}&\left(1-\chi e^{2\alpha x}\right)\frac{d^2V(x)}{dx^2}+2\left[(\alpha+\beta)-(2\alpha+\beta)\chi e^{2\alpha x}\right]\frac{dV(x)}{dx}\\&+\left\{-\beta(4\alpha+\beta)\chi e^{2\alpha x}+\left[\beta^2+2\alpha\beta+\left(\frac{\omega}{c}\right)^2\right]\right\}V(x)=0\end{aligned} \tag{8}$$

At this stage we make a choice of $\beta$ so that $\beta^2+2\alpha\beta+(\omega/c)^2=0$. Hence, $\beta_{1,2}=\alpha\left[-1\pm\sqrt{1-\left(\dfrac{\omega}{\alpha c}\right)^2}\right]$ and we make an arbitrary choice of the sign, so we assume

$$\beta = \beta(\alpha,\omega) = \alpha\left[-1+\sqrt{1-\left(\frac{\omega}{\alpha c}\right)^2}\right] \tag{9}$$

At the second step we change variable $x \to z$ so that $z = \chi e^{2\alpha x}$ and introduce function $W(z) = V\left[\frac{1}{2\alpha}\ln\left(\frac{z}{\mu}\right)\right]$. In the new variables equation (10) is represented as follows:

$$z(1-z)\frac{d^2W(z)}{dz^2} + \left[\left(2+\frac{\beta}{\alpha}\right) - \left(3+\frac{\beta}{\alpha}\right)z\right]\frac{dW(z)}{dz} - \frac{\beta}{\alpha}\left(1-\frac{\beta}{4\alpha}\right)W(z) = 0 \tag{10}$$

where $\beta$ is calculated by formula (9). Equation (10) could be rewritten in the standard Gauss hypergeometric equation form:

$$z(1-z)\frac{d^2W(z)}{dz^2} + \left[c-(a+b+1)z\right]\frac{dW(z)}{dz} - ab\cdot W(z) = 0 \tag{11}$$

where $a = \frac{\beta}{2\alpha} = \frac{1}{2}\left[-1+\sqrt{1-\left(\frac{\omega}{\alpha c}\right)^2}\right]$, $b = 2+\frac{\beta}{2\alpha} = \frac{1}{2}\left[3+\sqrt{1-\left(\frac{\omega}{\alpha c}\right)^2}\right]$ and $c = 1+\frac{\beta}{\alpha} = \left[1+\sqrt{1-\left(\frac{\omega}{\alpha c}\right)^2}\right]$.

Solution of equation (11) is

$$W(z) = \bar{C}_1 \cdot {}_2F_1(a,b;c;z) + \bar{C}_2 \cdot z^{1-c} \cdot {}_2F_1(b-c+1, a-c+1; 2-c; z) \tag{12}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function with parameters $a$, $b$, $c$ and argument $z$ and $\bar{C}_{1,2}$ are arbitrary constants.

In the original variables solution (12) could be rewritten as follows:

$$\begin{aligned} U(x) &= C_1 \cdot e^{-\alpha\left[1-\sqrt{1-\left(\frac{\omega}{\alpha c}\right)^2}\right]\cdot x} \cdot {}_2F_1\left(\frac{1}{2}\left[-1+\sqrt{1-\left(\frac{\omega}{\alpha c}\right)^2}\right], \frac{1}{2}\left[3+\sqrt{1-\left(\frac{\omega}{\alpha c}\right)^2}\right]; \left[1+\sqrt{1-\left(\frac{\omega}{\alpha c}\right)^2}\right]; \mu\cdot e^{2\alpha x}\right) \\ &+ C_2 \cdot e^{-\alpha\left[1+\sqrt{1-\left(\frac{\omega}{\alpha c}\right)^2}\right]\cdot x} \cdot {}_2F_1\left(\frac{1}{2}\left[3-\sqrt{1-\left(\frac{\omega}{\alpha c}\right)^2}\right], \frac{-1}{2}\left[1+\sqrt{1-\left(\frac{\omega}{\alpha c}\right)^2}\right]; \left[1-\sqrt{1-\left(\frac{\omega}{\alpha c}\right)^2}\right]; \mu\cdot e^{2\alpha x}\right) \end{aligned} \tag{13}$$

where $C_1 = \bar{C}_1$ and $C_2 = \chi^{1-c}\cdot\bar{C}_1$ are new arbitrary constants.

## 4. Computational scheme of the method of lines for the rod with variable cross-section

Let us return to equation (1) and rewrite it as follows:

$$\begin{aligned} &\rho S(x)\frac{\partial^2 u(x,t)}{\partial t^2} - \rho\eta^2\left[\frac{\partial I_p(x)}{\partial x}\frac{\partial^3 u(x,t)}{\partial t^2\,\partial x} + I_p(x)\frac{\partial^4 u(x,t)}{\partial t^2\,\partial x^2}\right] \\ &-E\left[\frac{\partial S(x)}{\partial x}\frac{\partial u(x,t)}{\partial x} + S(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right] = F(x,t) \end{aligned} \tag{14}$$

Next we divide the rod in $N+1$ equal intervals, so that $x_0 = 0$, $x_{N+1} = l$, and compose an approximate finite difference scheme for $x$ – differentiation at an arbitrary inner point $x_k$, $(k = 1,2,\cdots,N)$:

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \approx \frac{u_{k+1}-u_{k-1}}{2\cdot\Delta x}, \qquad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=x_k} \approx \frac{u_{k-1}-2\cdot u_{k+1}+u_{k+1}}{\Delta x^2} \tag{15}$$

where $\Delta x = \frac{l}{N+1}$ is length of the intervals of the rod.

Substituting (15) in (14) and regrouping terms we obtain the system of $N$ ordinary differential equations:

$$\begin{aligned}&-\left[J_k^{(2)}-J_k^{(1)}\right]\ddot{u}_{k-1}+\left(1+2J_k^{(2)}\right)\ddot{u}_k-\left[J_k^{(2)}+J_k^{(1)}\right]\ddot{u}_{k+1}\\&=-\left[J_k^{(3)}-J_k^{(4)}\right]u_{k-1}-\left[2J_k^{(4)}\right]u_k+\left[J_k^{(3)}+J_k^{(4)}\right]u_{k-1}+f_k(t)\end{aligned} \tag{16}$$

where $J_k^{(1)}=\frac{\eta^2\cdot dI_k}{2\cdot S_k\cdot\Delta x}$, $J_k^{(2)}=\frac{\eta^2\cdot I_k}{S_k\cdot\Delta x^2}$, $J_k^{(3)}=\frac{E\cdot dS_k}{2\cdot\rho\cdot S_k\cdot\Delta x}$, $J_k^{(4)}=\frac{E}{\rho\cdot\Delta x^2}$, $S_k=S(x_k)$, $I_k=I_p(x_k)$, $dS_k=\left.\frac{dS(x)}{dx}\right|_{x=x_k}$, $dI_k=\left.\frac{dI_p(x)}{dx}\right|_{x=x_k}$, $I_k=I_p(x_k)$ and $f_k(t)=\frac{1}{\rho\cdot S_k}F(t,x_k)$.

For the conical rod $S_k=\pi k^2\left(x_k-x_p\right)^2$ (remember that $x_p$ is the coordinate of the pole of the cone), $dS_k=2\pi k^2\left(x_k-x_p\right)$, $I_k=0.5\pi k^4\left(x_k-x_p\right)^4$, $dI_k=2\pi k^4\left(x_k-x_p\right)^3$,

$J_k^{(1)}=\frac{\eta^2\cdot k^2\cdot\left(x_k-x_p\right)}{\Delta x}$, $J_k^{(2)}=\frac{\eta^2\cdot k^2\left(x_k-x_p\right)^2}{2\cdot\Delta x^2}$, $J_k^{(3)}=\frac{E\cdot}{\rho\cdot\left(x_k-x_p\right)\cdot\Delta x}$ and $J_k^{(4)}=\frac{E}{\rho\cdot\Delta x^2}$.

For the exponential rod $S_k=\pi k^2e^{2\alpha x_k}$, $dS_k=2\alpha\pi k^2e^{2\alpha x_k}$, $I_k=0.5\pi k^4e^{4\alpha x_k}$, $dI_k=2\alpha\pi k^4e^{4\alpha x_k}$, $J_k^{(1)}=\frac{\eta^2\cdot k^2\cdot\alpha\cdot e^{2\alpha x_k}}{\Delta x}$, $J_k^{(2)}=\frac{\eta^2\cdot k^2\cdot e^{2\alpha x_k}}{2\cdot\Delta x^2}$, $J_k^{(3)}=\frac{E\cdot\alpha}{\rho\cdot\Delta x}$ and $J_k^{(4)}=\frac{E}{\rho\cdot\Delta x^2}$.

Unknowns $u_0=u(t,0)$ and $u_{N+1}=u(t,l)$ are defined from the boundary conditions. For example, for fixed ends $u_0=u_{N+1}=0$ and $\ddot{u}_0=\ddot{u}_{N+1}=0$. For free ends $\left.\frac{\partial u}{\partial x}\right|_{x=0}=0$ and (or) $\left.\frac{\partial u}{\partial x}\right|_{x=l}=0$. Derivatives at the end points are approximated as follows [ ]:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}\approx\frac{-3u_0+4u_1-u_2}{2\cdot\Delta x},\qquad\left.\frac{\partial u}{\partial x}\right|_{x=l}\approx\frac{u_{N-1}-4u_N+3u_{N+1}}{2\cdot\Delta x} \tag{17}$$

and hence, for free boundary conditions $u_0=\frac{4u_1-u_2}{3}$ (for $\left.\frac{\partial u}{\partial x}\right|_{x=0}=0$, and hence, $\ddot{u}_0=\frac{4\ddot{u}_1-\ddot{u}_2}{3}$) and (or) $u_{N+1}=\frac{4u_N-u_{N-1}}{3}$ (for $\left.\frac{\partial u}{\partial x}\right|_{x=l}=0$, and hence, $\ddot{u}_{N+1}=\frac{4\ddot{u}_N-\ddot{u}_{N-1}}{3}$). For different boundary conditions the corresponding values $u_0$, $u_{N+1}$ and $\ddot{u}_0$, $\ddot{u}_{N+1}$ could be estimated similarly.

## 5. Examples

For the conical Rayleigh-Love rod with fixed ends $\left(U(0)=U(l)=0\right)$ we obtain the following characteristic system of equations (see (6)):

$$D(\omega)=\det\left\|\begin{matrix}\dfrac{P_\sigma\left[-\dfrac{\eta k\omega}{c\sqrt{2}}x_p\right]}{\left(-x_p\right)} & \dfrac{Q_\sigma\left[-\dfrac{\eta k\omega}{c\sqrt{2}}x_p\right]}{\left(-x_p\right)}\\ \dfrac{P_\sigma\left[\dfrac{\eta k\omega}{c\sqrt{2}}\left(l-x_p\right)\right]}{\left(l-x_p\right)} & \dfrac{Q_\sigma\left[\dfrac{\eta k\omega}{c\sqrt{2}}\left(l-x_p\right)\right]}{\left(l-x_p\right)}\end{matrix}\right\|=0 \tag{18}$$

From this equation we calculate eigenvalues $\omega_n$ and eigenfunctions:

$$U_n(x)=\frac{P_\sigma\left[\frac{\eta k\omega_n}{c\sqrt{2}}(x-x_p)\right]\cdot Q_\sigma\left[-\frac{\eta k\omega_n}{c\sqrt{2}}x_p\right]-P_\sigma\left[-\frac{\eta k\omega_n}{c\sqrt{2}}x_p\right]\cdot Q_\sigma\left[\frac{\eta k\omega_n}{c\sqrt{2}}(x-x_p)\right]}{(x-x_p)\cdot Q_\sigma\left[-\frac{\eta k\omega_n}{c\sqrt{2}}x_p\right]} \tag{19}$$

Let us consider the conical rod with slope $k=0.1$. Its left end is fixed and located at $x_0=0$ m, right end is also fixed and located at $x_{N+1}=l=1$ m. The pole of the rod is located at $x_p=-0.5$ m. Modulus of elasticity of the rod is $E=100\cdot 10^9$ Pa, mass density $\rho=8.5\cdot 10^3$ kg/m$^3$ and Poisson ratio is $\eta\approx 0.33$ (for calculation the Poisson ratio was taken with eight digits after coma as $\eta=0.33296357$ because at this value $\sigma=-\frac{1}{2}+\sqrt{\frac{9}{4}+\frac{2}{(\eta k)^2}}\approx 42+1.161\cdot 10^{-6}$ is very close to integer value $\sigma=42$, which substantially simplified calculations of the Legendre functions $P_\sigma(z)$ and $Q_\sigma(z)$). Simulation of the problem was performed in MATHCAD14 which has the built-in function $Leg(\sigma,x)$ for calculation of $P_\sigma(z)$ with integer $\sigma$. Function $Q_\sigma(z)$ with integer $\sigma$ calculated as follows[4, 5]:

$$Q_\sigma(z)=\frac{1}{2}P_\sigma(z)\ln\left(\frac{1+z}{1-z}\right)-\sum_{q=1}^{\sigma}\left[\frac{1}{q}P_{q-1}(z)P_{\sigma-q}(z)\right]$$

Distribution of eigenvalues of the problem (equation (18)) is shown in Fig. 1 (solid line) where it is compared with the eigenvalues distribution of the rod with the same geometric and physical properties but considered in the frames of the classical theory (dotted line).The eigenvalues considered in the frames of the Rayleigh-Love theory have the limiting point which in this case is approximately equal to 15.438 kHz. Eigenfunctions corresponding to the first five eigenvalues are shown in Fig. 2. These eigenfunctions were plotted using exact solution (6).

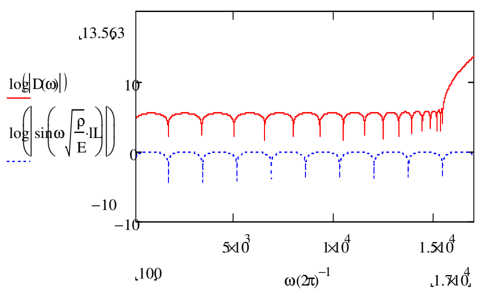


**Figure 1.** Eigenvalues of the Rayleigh Love (solid red line) and classical (dotted blue line) conical rods.

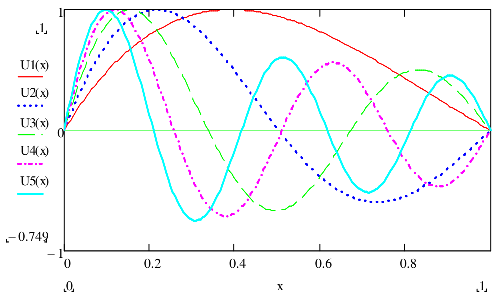


**Figure 2.** First five eigenfunctions of the Rayleigh-Love conical rod.

Let us consider free vibrations of the Rayleigh-Love conical rod at $F(x,t)=0$, corresponding to initial conditions $u(x,t)\big|_{t=0}=g(x)$, $\frac{\partial u(x,t)}{\partial t}\Big|_{t=0}=0$. The analysis was performed by means of expressions (20) - (21) and by means of the method of lines in which the conical rod was divided in $N+1=101$ equal intervals and numerical integration of the system of $N=100$ ordinary differential equations was performed by the Adams-backward differentiation formula method with tolerance $10^{-15}$. All solutions gave the similar results which are shown in Fig. 3 – 6. In Fig. 3 we assumed that initial condition is proportional to the first eigenfunction $g(x)=10^{-3}U_1(x)$ (see Fig. 2), the time integration was performed in interval $t \in [0, 2 \cdot T_1]$ seconds, where $T_1 = 2\pi / \omega_1$ and $\omega_1$ is the first eigenvalue. Time interval $2 \cdot T_1$ is subdivided into 1000 intervals. The Fourier analysis of the time realization shown that absolute difference between the exact eigenvalue and eigenvalue calculated by the method of lines is $\Delta f_1 = 0.069$ Hz which corresponds to $(\Delta f_1 / f_1) \cdot 100\% = 4 \cdot 10^{-3}\,\%$. $(\Delta f_2 / f_2) \cdot 100\% = 1.6 \cdot 10^{-2}\,\%$. For $N+1 = 201$ intervals the results of solution of the system of $N = 200$ ordinary differential equation are $\Delta f_1 = 0.017$ Hz and $(\Delta f_1 / f_1) \cdot 100\% = 1.0 \cdot 10^{-3}\,\%$. In Fig. 4 the initial condition were taken proportional to the second eigenfunction $g(x) = 10^{-3} U_2(x)$ (Fig. 2), the time integration was performed in interval $t \in [0, 2 \cdot T_2]$ seconds, where $T_2 = \dfrac{2\pi}{\omega_2}$ and $\omega_2$ is the second eigenvalue. Results of the Fourier analysis of the time realization shown that absolute difference between the exact eigenvalue and eigenvalue calculated by the method of lines is $\Delta f_2 = 0.54$ Hz which corresponds to $(\Delta f_2 / f_2) \cdot 100\% = 1.6 \cdot 10^{-2}\,\%$. For $N+1 = 201$ intervals the results of solution of the system of $N = 200$ ordinary differential equation are $\Delta f_2 = 0.135$ Hz and $(\Delta f_2 / f_2) \cdot 100\% = 4 \cdot 10^{-3}\,\%$.

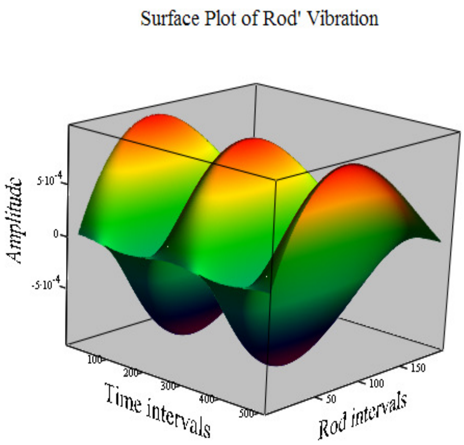


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**Figure 3.** Free vibrations of the Rayleigh-Love conical rod at the first mode.

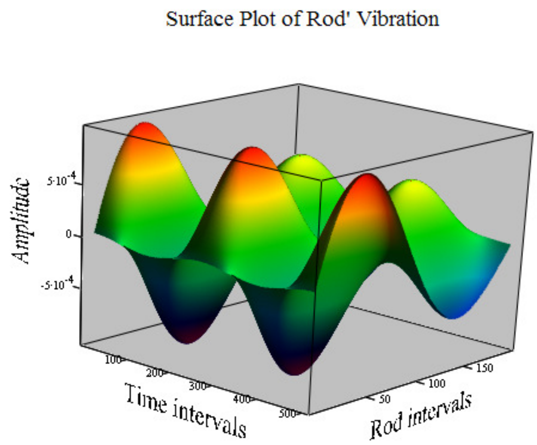


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**Figure 4.** Free vibrations of the Rayleigh-Love conical rod at the second mode.

In Fig. 5 the initial condition were taken proportional to the second eigenfunction $g(x) = 10^{-3} U_3(x)$ (Fig. 2), the time integration was performed in the time interval $t \in [0, 2 \cdot T_3]$ seconds, where $T_3 = \dfrac{2\pi}{\omega_3}$ and $\omega_3$ is the third eigenvalue. Results of the Fourier analysis of the time realization shown that absolute difference between the exact eigenvalue and eigenvalue calculated by the method of lines is $\Delta f_3 = 1.75$ Hz which corresponds to $(\Delta f_3 / f_3) \cdot 100\% = 3.5 \cdot 10^{-2}\,\%$. For $N+1 = 201$ intervals the results of solution of the system of $N = 200$ ordinary differential equation are $\Delta f_3 = 0.437$ Hz and $(\Delta f_3 / f_3) \cdot 100\% = 9 \cdot 10^{-3}\,\%$. In Fig. 6 the initial condition were taken proportional to the second eigenfunction $g(x) = 10^{-3} U_4(x)$ (Fig. 2), the time integration was performed in the time interval $t \in [0, 2 \cdot T_4]$ seconds, where $T_4 = \dfrac{2\pi}{\omega_4}$ and $\omega_4$ is the fourth eigenvalue. Results of the Fourier analysis of the time realization shown that absolute difference between the exact eigenvalue and eigenvalue calculated by the method of lines is $\Delta f_4 = 3.92$ Hz which corresponds to $(\Delta f_4 / f_4)\cdot 100\% = 6\cdot 10^{-2}\,\%$. For $N+1 = 201$ intervals the results of solution of the system of $N = 200$ ordinary differential equation are $\Delta f_4 = 0.975$ Hz and $(\Delta f_4 / f_4)\cdot 100\% = 1.5\cdot 10^{-2}\,\%$.

One can see that the results of numerical simulation by the method of lines are very close to the theoretically predicted results. Accuracy of estimations is increasing with increasing of the number of intervals of the rod's length. Hence, we can conclude that the method of line is a reliable numerical method of simulation of partial differential equations with mixed time-spatial derivatives.

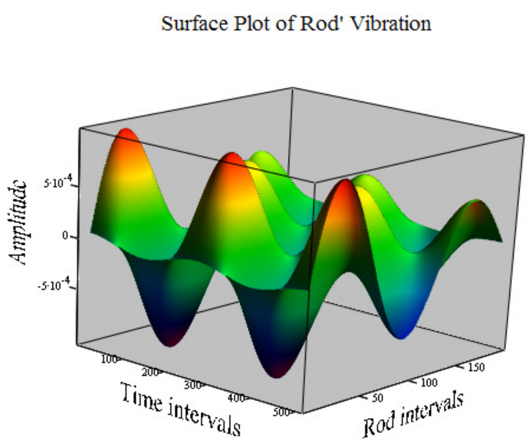


**Figure 5.** Free vibrations of the Rayleigh-Love conical rod at the third mode.

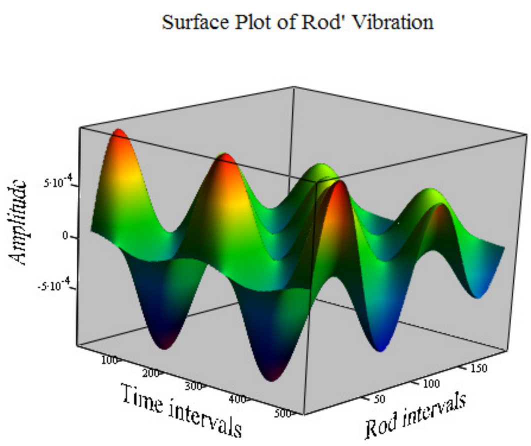


**Figure 6.** Free vibrations of the Rayleigh-Love conical rod at the fourth mode.

## 6. Conclusions

Two exact solutions of equations of motion were derived for the case of longitudinal vibrations of the Rayleigh-Love rod. The first exact solution was obtained for the conical rod and expressed in the Legendre functions. The second exact solution was obtained for the exponential rod and expressed in the Gauss hypergeometric functions. The general solutions of the problem are formulated in terms of two alternative Green functions. The computational scheme of the method of lines was formulated for the case of the Rayleigh-Love rod with variable cross-section. Solutions obtained by the method of lines for the conical rod were compared with the exact solutions of the problem. It was shown that the method of lines produces results which are very close to the corresponding exact solutions. It was also shown that the accuracy of the method of lines is increasing with increasing of number of intervals on the rod. The conclusion was formulated that the method of lines generates reliable and accurate results for partial differential equations with mixed time-spatial derivatives.

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