One- and two-dimensional topological charge distributions in stochastic optical fields

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**Statistical approach**

It is **not** possible to formulate a general theory that can predict vortex trajectories $x_n(z)$ from arbitrary initial vortex parameters.

**Reason**: the vortex degrees of freedom are inseparable from other degrees of freedom in optical beams.

However:

- Vortex dynamics may be predictable in a **statistical** sense.
- Quantities would be defined in terms of probability distributions.
- Justification: the other degrees of freedom average out.
- Different perspective in terms of the kind of questions that are addressed.
Definitions

Vortex number density: Number of vortices per cross-section area.
→ function of transverse coordinates \((x, y)\) that can changes as a function of propagation distance \(z\)

- Positive vortex density \(n_p(x, y, z) \geq 0\)
- Negative vortex density \(n_n(x, y, z) \geq 0\)
- Combined vortex density \(V(x, y, z) = n_p(x, y, z) + n_n(x, y, z) \geq 0\)
- Topological charge density \(T(x, y, z) = n_p(x, y, z) - n_n(x, y, z)\)
Speckle fields

Speckle field contains a random vortex field in equilibrium

- Globally: neutral topological charge
  (⇔ adjacent topological charges are anti-correlated)
- Annihilation rate = creation rate (⇒ equilibrium!)
- Equilibrium vortex density is determined by the properties of the speckle field

\[ V_{eq} = -\frac{C''_{x=0}}{4\pi} = \frac{A_c}{2} \]

\( A_c \) — coherence area
\( C \) — autocorrelation function

Topological charge density

Analytic calculation\(^a\)

\[
T_A = \frac{1}{A} \int_A \delta(\psi_r) \delta(\psi_i) \left( \partial_x \psi_r \partial_y \psi_i - \partial_x \psi_i \partial_y \psi_r \right) \, dx \, dy
\]

\[
T(x) = \int \exp \left( -Q^\dagger M^{-1} Q \right) \frac{1}{\pi^3 \det(M)} (q_3 q_6 - q_5 q_4) \, d^4 q \bigg|_{q_1=q_2=0}
\]

with

\[
M = \begin{bmatrix}
\langle \psi \psi^* \rangle & \langle \psi_x \psi^* \rangle & \langle \psi_y \psi^* \rangle \\
\langle \psi \psi^*_x \rangle & \langle \psi_x \psi_x^* \rangle & \langle \psi_y \psi_x^* \rangle \\
\langle \psi \psi^*_y \rangle & \langle \psi_x \psi_y^* \rangle & \langle \psi_y \psi_y^* \rangle
\end{bmatrix}
\]

and

\[
Q = \begin{bmatrix}
q_1 + iq_2 \\
q_3 + iq_4 \\
q_5 + iq_6
\end{bmatrix}
\]

\[
T(x) = \frac{i \left( \langle \psi_y \psi_x^* \rangle - \langle \psi_x \psi_y^* \rangle \right)}{2\pi \langle \psi \psi^* \rangle} + \frac{i \left( \langle \psi \psi^*_y \rangle \langle \psi_x \psi^* \rangle - \langle \psi_y \psi^* \rangle \langle \psi_x \psi_x^* \rangle \right)}{2\pi \langle \psi \psi^* \rangle^2}
\]

\(^a\)MV Berry and MR Dennis, *Proc. R. Soc. London A* 456, 2059-2079 (2000);

1D inhomogeneous fields

Experimental setup:

\[ \psi_{in} = \tilde{\psi}_1 \sin(\alpha_xx) \exp(-i\alpha_yy) + \tilde{\psi}_2 \cos(\alpha_xx) \exp(i\alpha_yy) \]

Numerical simulation:

Beam propagation \(\rightarrow\) extract vortex distribution
1D topological charge density

Analytical result:

\[ T(x, z) = \frac{\alpha_x \alpha_y}{\pi} \sin(2\alpha_x x) \exp\left(-\frac{1}{2} \lambda^2 \alpha_x^2 W^2 z^2\right) \]

Definition: \( \alpha_x = 2\pi a_0 \) and \( \alpha_y = 2\pi b_0 \)

Comparison with numerical results: \( (a_0 = 2, b_0 = 16, W = 32) \)
1D topological charge density

Evolusion of topological charge density with input produced by direct phase modulation (SLM or DOE)

Numerical results:

\[ \partial_z T - z\kappa_0 \nabla^2 T = 0 \]

\[ \kappa_0 = \frac{\lambda^2}{\pi d^2} \]

\[ d = \text{coherence length} \]

\[ a \]

1D dynamics

Analytic result

\[ T(x, z) = \frac{\alpha_x \alpha_y}{\pi} \sin(2\alpha_x x) \exp \left( -\frac{1}{2} \lambda^2 \alpha_x^2 W^2 z^2 \right) \]

is a solution of: \[ \partial_z T - z\kappa_0 \nabla^2 T = 0, \]

with \[ \kappa_0 = \frac{1}{4} \lambda^2 W^2 \quad \Rightarrow \quad d^2 = A_c = \frac{1}{\pi W^2} \]

which is consistent with the definitions of the equilibrium vortex charge
Experimental setup is a generalization of the 1D case:

\[
\psi_{in} = \tilde{\psi}_1 \cos(\beta_x x) \cos(\beta_y y) \exp[-i(\alpha_x x + \alpha_y y)] \\
+ \tilde{\psi}_2 \cos(\beta_x x) \sin(\beta_y y) \exp[i(\alpha_x x - \alpha_y y)] \\
+ \tilde{\psi}_3 \sin(\beta_x x) \cos(\beta_y y) \exp[-i(\alpha_x x - \alpha_y y)] \\
+ \tilde{\psi}_4 \sin(\beta_x x) \sin(\beta_y y) \exp[i(\alpha_x x + \alpha_y y)]
\]
Analytical result:

\[ T(x) = \frac{f_1(z) \sin(2\beta_xx) + f_2(z) \sin(2\beta_yy) + f_3(z) \cos(2\beta_xx) \cos(2\beta_yy)}{\pi[1 + f_4(z) \sin(2\beta_xx) \sin(2\beta_yy)]^2} \]

where

\[ f_1(z) = \frac{1}{2} \alpha_x \beta_y \exp[-z^2 \eta(\beta_x^2 + 2\beta_y^2)] \sin(2zK_y) \sin(zK_x) \]
\[ - \alpha_y \beta_x \exp(-\eta \beta_x^2 z^2) \cos(zK_x) \]

\[ f_2(z) = -\frac{1}{2} \alpha_y \beta_x \exp[-z^2 \eta(\beta_y^2 + 2\beta_x^2)] \sin(2zK_x) \sin(zK_y) \]
\[ + \alpha_x \beta_y \exp(-\eta \beta_y^2 z^2) \cos(zK_y) \]

\[ f_3(z) = \exp[-z^2 \eta(\beta_x^2 + \beta_y^2)] \left[ \alpha_y \beta_x \sin(zK_x) \cos(zK_y) \right. \]
\[ - \alpha_x \beta_y \sin(zK_y) \cos(zK_x) \left] \right] \]

\[ f_4(z) = \exp[-z^2 \eta(\beta_x^2 + \beta_y^2)] \sin(zK_y) \sin(zK_x) \]

\[ K_x = \frac{\lambda \alpha_x \beta_x}{\pi} \quad K_y = \frac{\lambda \alpha_y \beta_y}{\pi} \quad \eta = \frac{\lambda^2 W^2}{2} \]
Comparison with numerical results

Definitions: $\alpha_x = 2\pi a_x$, $\alpha_y = 2\pi a_y$, $\beta_x = 2\pi b_x$ and $\beta_y = 2\pi b_y$

Parameters: $a_x = 2$, $a_y = 4$, $b_x = 16$, $b_y = 32$

$W = 32$

$W = 16$
Topological charge evolution
Phase drift

Topological charge $\rightarrow$ phase slope $\rightarrow$ sideways drift

$$\Delta x \approx -\frac{\nabla \theta \Delta z}{k} \quad \text{for} \quad k \gg |\nabla \theta|$$

for $\Delta z \rightarrow 0$:

$$\partial_z T(x, z) = \frac{\nabla \theta \cdot \nabla T(x, z)}{k}$$

Gradient of the phase function: $\nabla \theta = T(x, z) \ast \nabla \phi$

Drift term:

$$\partial_z T(x, z) = \frac{1}{k} \left[ T(x, z) \ast \nabla \phi \right] \cdot \nabla T(x, z)$$

where $\ast =$ convolution and

$$\nabla \phi(x, y) = \frac{y \hat{x} - x \hat{y}}{x^2 + y^2}$$
Using combination of speckle fields one can produce inhomogeneous vortex distributions that allow both analytical calculations and numerical simulations.

One-dimensional topological charge density:
- Gaussian decay obeys (modified) diffusion equation
- Diffusion parameter is related to coherence area

Two-dimensional topological charge density:
- The same diffusion behaviour
- Additional nonlinear behaviour may be explained by drift mechanisms