Characterisation of the flow regimes of arbitrary manoeuvre in absolute and relative frames

I. Gledhill and J. Nordström

1 Aeronautical Systems Competency Area, Defence, Peace, Safety and Security Operational Unit, CSIR
2 Dept of Mathematics, U Linköping, Linköping, Sweden

Outline

Warning: this is a theory paper

• Context
• Background
• Some of the CFD background
• Computational background
• Theoretical background
• In the relative frame, in order to investigate relative size of terms,
• Continuity equation
• Momentum equation
• Energy equation

Outline II

• Dimensionless constants
• A question: longitudinal independence in rockets?
• Some illustrations
• Conclusions
• Further work

Context

• Engineering tools useless without…
• Engineering judgment which is based on…
• Understanding

Objectives: overall

• A formal framework for arbitrary manoeuvre
• CFD modelling of arbitrary manoeuvre
• Characterise dynamic loads in arbitrary manoeuvre

Objectives: specific

• Include acceleration terms \( \dot{r} \) and \( \dot{\omega} \) in relative (body) frame formulation
• Include energy equation
• Find dimensionless numbers that are useful
• And thereby build the next step in the programme

Note: can only characterise linear behaviour in this way; nonlinear behaviour needs models

Background

• Batchelor, Greenspan, Landau and Lifshitz
• Directed largely at understanding atmospheric flows and waves
• Axial turbines
• Rothapy and constant \( \omega \) behaviour
• Flight dynamics
• The aims: bring this old news into CFD of arbitrary manoeuvres
**CFD background**

- Moving grids:
  - Chimera overset grids
  - Arbitrary Lagrangian-Eulerian, ALE
  - Constant rotation w: turbines and compressors
  - Small perturbations: aerelasticity
  - Small perturbations: dynamic derivatives

- In the inertial frame:
  - Rosihan et al. 2007..2011

**Theory Background to present work**

- Transformation between frames moving with constant relative velocity is trivial: Galilean
  - Transformation between frames with relative acceleration is subject of present programme

- Löfgren
  - General formulation of transforms in 4-space between inertial and relative frames
  - Invariants in transformation

- Forsberg
  - Löfgren’s formulation to simpler formulation
  - Numerical implications of inertial and relative frames
  - Stability and convergence in inertial and relative frames

- Forsberg et al. 2009
  - Implementation, validation and test cases

**The parameters**

\[ \Sigma \text{ inertial frame: absolute} \]

- Position vector of fluid element
- Fluid velocity
- Position of \( O' \)
- Velocity of \( O' \)
- Rotation vector of \( \Sigma' \) relative to \( \Sigma \)
- Rotational transform \( \chi = r + U \cdot \omega \)

**Vectors and transforms**

\[ \Sigma \text{ inertial frame: absolute} \]

\[ \Sigma' \text{ body frame: absolute} \]

- Position vector of fluid element
- Fluid velocity
- Position of \( O' \)
- Velocity of \( O' \)

Transform
\[ \frac{\partial}{\partial \chi} = \frac{\partial}{\partial \chi'} \]

**Vectors interpreted in the other frame**

- \( \vec{a} \) as seen in \( \Sigma' \) is interpreted as \( \vec{\alpha} \)
- \( \vec{x} \) as seen in \( \Sigma' \) is interpreted as \( \vec{\alpha} \)
- \( \vec{\alpha} \) as seen in \( \Sigma' \) is interpreted as \( \vec{\alpha} \)

**Gravity vector interpreted in the other frame**

- \( g \) as seen in \( \Sigma' \) is interpreted as \( \vec{g} \)
Scalars and intrinsic variables

Absolute, inertial
- \( \rho \)
- \( p \)
- \( T \)
- \( \mu \)
- \( \nu \)
- \( k \)
- \( \sigma \)
- \( \tau \)

Stress tensors are dependant on velocity

Relative: Conserved
- \( \rho \)
- \( p \)
- \( T \)
- \( \mu \)
- \( \nu \)
- \( k \)
- \( \sigma \)
- \( \tau \)

Is entropy \( S \) conserved?

Why is this notation so complex?
- It distinguishes in detail the frame transforms
- And provides a general framework

The general equation in conservation form for a conserved intrinsic quality \( a \)

Relative
\[
\frac{\partial}{\partial t} (\rho a) + \nabla \cdot (\rho \mathbf{u} a) = \dot{Q}_a
\]

Mass conservation: the equation of continuity

Relative
\[
\frac{\partial}{\partial t} (\rho p) + \nabla \cdot (\rho \mathbf{u} p) = 0
\]

Momentum conservation

Relative
\[
\frac{\partial}{\partial t} (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I}) = \dot{\mathbf{\tau}} - \rho \mathbf{a}_w - 2 \rho \mathbf{\alpha} \mathbf{u} - \rho \mathbf{\alpha} \otimes \mathbf{u} - \rho \mathbf{\alpha} \otimes \mathbf{\alpha}
\]

Energy conservation

Relative
\[
\dot{E} = \frac{1}{2} |\mathbf{\dot{u}}|^2 + \dot{\mathbf{\tau}} \cdot \mathbf{\tau}
\]
\[
\frac{\partial}{\partial t} (\rho E) + \nabla \cdot (\rho \mathbf{u} E + \rho \mathbf{u} - \mathbf{\tau} \cdot \mathbf{u} + \rho \mathbf{\alpha} \cdot \mathbf{u} + \rho \mathbf{\alpha} \cdot \mathbf{\alpha}) = \dot{q}_w + \rho \mathbf{a}_w \cdot \mathbf{\dot{u}} + \rho \mathbf{\alpha} \cdot \mathbf{\dot{u}} + \rho \mathbf{\alpha} \cdot \mathbf{\alpha}
\]
1. First objective achieved…

- Write out the equations including $g$ and Viscous effects;
- And write out the energy equation in the relative frame.

Why?
- Now we can look at physical effects.

2. Next: find useful dimensionless constants for the momentum equation

- Several assumptions and notes:
  - First: comparison to convective effects
  - Only linear effects are identified this way
  - For the present, single-scale problems are written; but most scales will differ from $x$ scales
  - FYSA, Therefore, some gross simplifications

Momentum equation: typical scales

- $\frac{\partial}{\partial t} (\rho u) + \nabla \cdot (\rho u u) = -\rho g + \rho \frac{\partial u}{\partial x} - 2 \rho \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + p \frac{\partial^2 u}{\partial x^2} + f$

- Strouhal

The formal structure underlying the simulations of Roohani and Skews 2007…

$\frac{L}{U} \frac{\partial}{\partial t} (\rho u) + \nabla \cdot (\rho u u) = -\rho g + \rho \frac{\partial u}{\partial x} - 2 \rho \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + p \frac{\partial^2 u}{\partial x^2} + f$

Extract meaning from models, term by term Strouhal

$\frac{L}{U} \frac{\partial}{\partial t} (\rho u) + \nabla \cdot (\rho u u) = -\rho g + \rho \frac{\partial u}{\partial x} - 2 \rho \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + p \frac{\partial^2 u}{\partial x^2} + f$

- Strouhal number: typical temporal behaviour $St^* = \frac{L}{U}$
Euler number

\[
E_n = - \frac{\rho L}{\mu}
\]

Reynolds: translational viscous effects

\[
E_n = - \frac{\rho L}{\mu}
\]

Ekman: rotational viscous effects

\[
E_n = - \frac{\rho L}{\mu}
\]

Ekman number: development of boundary layers and end-wall viscous phenomena in rotationally dominated flows

\[
E_k = \frac{V}{\Omega}
\]

Translational Acceleration of the frame

\[
\frac{\partial}{\partial t} \left( \frac{\rho V^2}{2} \right) + \nabla \cdot \left( \rho V V \right) = \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \right) \nabla \cdot \left( \frac{1}{\rho} \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right)
\]

Translation: Coriolis effects

\[
\frac{\partial}{\partial t} \left( \frac{\rho V^2}{2} \right) + \nabla \cdot \left( \rho V V \right) = \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \right) \nabla \cdot \left( \frac{1}{\rho} \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right)
\]

Coriolis effects

\[
\frac{\partial}{\partial t} \left( \frac{\rho V^2}{2} \right) + \nabla \cdot \left( \rho V V \right) = \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \right) \nabla \cdot \left( \frac{1}{\rho} \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right)
\]

Rotational Acceleration of the frame

\[
\frac{\partial}{\partial t} \left( \frac{\rho V^2}{2} \right) + \nabla \cdot \left( \rho V V \right) = \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \right) \nabla \cdot \left( \frac{1}{\rho} \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right)
\]

Angular acceleration

\[
\frac{\partial}{\partial t} \left( \frac{\rho V^2}{2} \right) + \nabla \cdot \left( \rho V V \right) = \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \right) \nabla \cdot \left( \frac{1}{\rho} \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right)
\]

Angular acceleration

\[
\frac{\partial}{\partial t} \left( \frac{\rho V^2}{2} \right) + \nabla \cdot \left( \rho V V \right) = \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \right) \nabla \cdot \left( \frac{1}{\rho} \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right) + \frac{\partial}{\partial t} \left( \frac{\mu}{\nu} \mathbf{V} \cdot \nabla \cdot \mathbf{V} \right)
\]
Centrifugal effects

\[ \frac{\partial}{\partial t} \left( \frac{\rho u^2}{2} \right) + \nabla \cdot \left( \rho u u \right) + \nabla \cdot \left( \rho u u \times \Omega \right) \cdot \nabla \phi = \frac{\partial}{\partial t} \left( \frac{\rho u u^2}{2} \right) - \nabla \cdot \left( \rho u u \times \Omega \right) \cdot \nabla \phi \]

Gravitational effects

\[ \frac{\partial}{\partial t} \left( \frac{\rho g}{2} \right) + \nabla \cdot \left( \rho u g \right) + \nabla \cdot \left( \rho u \phi \right) \cdot \nabla \phi = \frac{\partial}{\partial t} \left( \frac{\rho g}{2} \right) + \nabla \cdot \left( \rho u \phi \right) \cdot \nabla \phi \]

- Buoyancy term in comparison to pressure effects in hydrostatics

3. A question: can Taylor columns exist in missile combustion chambers?
- When do rotational effects dominate?

Taylor-Proudman theorem
- Constant angular velocity \( \Omega \)
- Rossby number \( Ro = \frac{U}{2\Omega L} \)
- For \( Ro < 1 \), rotational effects dominate convection
- For incompressible, inviscid flow,
  \( \frac{\partial}{\partial z} = 0 \)
- An obstacle \( A \) which is moved and generates streamlines at \( z_0 \)
- ...generates identical streamlines at all \( z \)
- Could Taylor columns exist in internal flow in missiles and rockets?
- What influence can be predicted on external flow?
- Progressive assumptions

- Accomplished by slight spin-up or spin-down
  - Theory, Proudman 1916
  - Experiment, Taylor 1917
Rockets or missiles considered as…

- Closed cylinders,
- Incompressible,
- Almost rigid rotation,
- With Rossby number Ro characterising rotational dominance
- Viscous effects can be characterised [Greenspan] by times scales related to the
  Ekman number:
  1 \equiv 1, development of viscous boundary layers (I),
  1 \equiv \Omega \mathcal{E}k^{-1/2}, spin-up time (II, III),
  1 \equiv \Omega \mathcal{E}k^{-1/2}, decay of residual viscous effects
- Take \nu for dry air for the present

Typical values

- For rigid rotation, Ro \approx \frac{1}{2}
- Will viscous effects dominate Taylor columns?
  - Apache and Cajun sounding rockets
  - Uselton and Carman ADC 1969
    - \Omega \approx 1.5 to 32 rads^{-1}, radius r \approx 59 mm
  - Highly manoeuvrable missiles,
    - Marquardt, Lawrence and Lawrence, AEDC, 1998
      - \Omega \approx 100 rads^{-1}, radius r \approx 59 mm
  - Unguided fin-stabilised artillery rockets, 122mm
  - Khalil et al., Egypt Armed Forces, 2009
    - \Omega \approx 100 rads^{-1}, radius r \approx 59 mm

Angular acceleration and translational acceleration

- Khalil et al. 2009
- Fin stabilised artillery rockets
- Measurements of translational acceleration, spin rate available
  - \frac{\Delta \Omega}{\Delta t} \approx 100 rads^{-1}, \omega = 100 rads^{-1}
  - \frac{\Delta U}{\Delta t} \approx \frac{1}{150000}
- Linear approximation indicates low angular acceleration effects…
- But we are aware of the vortex interactions through CFD
  - \frac{\Delta U}{\Delta t} \approx 500 ms^{-2} for thrust of 23600N,
  - \frac{\Delta U}{\Delta t} \approx 2.1, a significant factor even in these terms:
    translational acceleration likely to have significant effect

Translational acceleration
Roohani and Skews, 2007 and 2011

- Biconvex, NACA0012, NACA2412 and RAE2812 airfoils
  - \Delta T = 1041ms^{-2}
    For transonic cases,
    - \frac{\Delta L}{\Delta t} \approx 0.012
      But very significant changes are experienced – and these are non-linear, due to shock position and shape
    - Subsonic cases: U\approx 100 ms^{-1}
      - \frac{\Delta L}{\Delta t} \approx 0.1, and significant changes in linear range should be apparent

Conclusions

1. the energy equation is expressed in terms of translations and rotational acceleration,
2. the generalised enthalpy equation is similarly expressed,
3. but the meaning of these is still to be explored
4. dimensionless constants for momentum changes are re-derived,
5. But translational acceleration needs reconsideration in the light of numerical experiments,
6. And rotational indications by Ro and Ek are that Taylor columns in rockets need consideration in terms of heat transfer, boundary conditions and compressibility
Further work

- Compressibility
- Thermodynamics: is entropy conserved in frame transformation?
  - Express energy equation in terms of T
  - Derive dimensionless numbers
- Taylor columns:
  - Compressibility
  - Boundary conditions
  - Heat transfer
- Shocks:
  - Do Rankine-Hugoniot relations transform?

Further work II

- Boundary conditions on accelerating walls
  - Boundary layer formation
- Turbulence
  - How do we deal with numerical turbulence models?
  - Is it appropriate to apply classic turbulence models even in the absolute frame?
- Perturbations
  - Rossby waves
  - Orr-Sommerfeld and transition

¿questions?